

THE BULLETIN OF THE



USER GROUP

**+ CAS-TI**

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Hello Josef,

I just learned about

<http://www.ibiblio.org/technical/packages/mathtools/MathTools.pdf>

which you might already know about.

It certainly deserves publicity in your next newsletter if you haven't already done so.

-- best wishes for you and your family.

David Stoutemyer

This is the TechniCalc website:

<http://www.ibiblio.org/technical/>

<h1 style="margin: 0;">TechniCalc</h1> <p style="margin: 0;"><b>Handheld computing in Math, Science, and Engineering</b></p> <p style="margin: 0; font-size: small;">TechniCalc is an alliance of sites devoted to the use of handheld computers and calculators in industry and academics.</p> <p style="margin: 0; font-weight: bold; font-size: small;">Member sites:</p>		
<p style="margin: 0; font-size: small;"><a href="#">TI-89/TI-92+ Examples</a> Andrew Cacovean's site contains several tutorials designed to illustrate the use of the TI-89 or TI-92+ calculators in solving various mathematical/engineering problems.</p>		
<h2 style="margin: 0; color: gold;">TI Graphing Calculator FAQ</h2> <p style="margin: 0; font-size: small;">Ray Kremer's FAQ covers all popular models of TI graphing calculators with information on math tasks, games, and more.</p>		
<h2 style="margin: 0; color: purple;">FRANK P. WESTLAKE</h2> <p style="margin: 0; font-size: small;">This archives programs made by Frank Westlake for the TI-89 family of graphing calculators.</p>		
<p style="margin: 0; font-size: small;"><a href="#">Bhuvanesh Bhatt's TI-89/TI-92 Plus/Voyage 200 page</a> Bhuvanesh Bhatt's page has many resources for the TI-68k family of graphing calculators, including the MathTools package and the free EQW equation writer.</p>		
<div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 0 auto;"> <p style="margin: 0; font-size: small;">TI-TipList</p> </div> <p style="margin: 0; font-size: small;">Doug Burkett's Tip List is the largest single collection of tips, tricks and helpful examples for the TI-89, TI-92 and Voyage 200 calculator families.</p>		
<p style="margin: 0; font-size: small;"><a href="#">TechniCalc Forums</a> Discuss TechniCalc member sites here</p>		
<table style="margin: 0 auto; font-size: x-small;"> <tr> <td style="vertical-align: top;"> <p><b>Other forums we like:</b>  <a href="#">TI's discussion groups</a>  <a href="#">United-TI's forum</a>  <a href="#">comp.sys.hp48</a></p> </td> <td style="vertical-align: top;"> <p><b>More links:</b>  <a href="#">Texas Instruments calculators</a>  <a href="#">ticalc.org</a>  <a href="#">Hewlett-Packard calculators</a>  <a href="#">hpcalc.org</a></p> </td> </tr> </table>	<p><b>Other forums we like:</b>  <a href="#">TI's discussion groups</a>  <a href="#">United-TI's forum</a>  <a href="#">comp.sys.hp48</a></p>	<p><b>More links:</b>  <a href="#">Texas Instruments calculators</a>  <a href="#">ticalc.org</a>  <a href="#">Hewlett-Packard calculators</a>  <a href="#">hpcalc.org</a></p>
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I downloaded Mathtools.pdf and the respective TI-92/V200 grouped file and sent a mail to Bhuvanesh Bhatt that I would like to prepare his library (347 functions from agm to zetaprim !!!) for the TI-NspireCAS platform. He replied very friendly:

Dear Professor Boehm,

Please feel free to port any of my programs to the TI-Nspire. I myself don't have that device, but assuming the operating system is similar to the TI-68k series (TI-89, TI-89 Titanium, TI-92/92+, V200) I might be able to help with the port.

Regards,  
Bhuvanesh

See the first function of the TI-NspireCAS mathtool-library on page 48.

Dear DUG Members,

This was indeed a memorable and strange spring time. A wee bite changed our all lives. "C" finds its expression in this DNL, of course. Sebastian Rauh's simulation of the spread and my modelling the development of the infection in Austria is our respective contribution. Sebastian was very productive and provided tools for capturing data from the TI-Nspire Graphs application.

Wolfgang Alvermann sent his mathematics final exam from 1968 together with his treatment how it has been usual in that times. I could not resist to add the TI-Nspire solutions with a few add-ons. I sent my solution to Wolfgang and he answered: "[Inspecting your Nspire elaboration I am very happy that we in that time had to work manually.](#)"

It might be charming if some of you would also find their examination tasks from more or less time ago. We could compare with recent task with or without technology use permitted.

Enjoy like me Rick Nungester's colourful tubes and his function to produce them. This is a significant improvement compared with the SPACE\_TUBE-function provided in DfW5's utility file GRAPH.MTH. His function could easily be ported to the TI-Nspire platform.

Many thanks for your mails, ideas and cooperation and until next time with my best regards

Yours Josef

DUG-Mitglieder,

Wir haben doch ein denk- und merkwürdiges Frühjahr hinter uns. Ein winziges Etwas hat unser aller Leben umgestaltet. „C“ findet auch in diesem DNL seinen Niederschlag. Sebastian Rauhs Simulation und meine Modellierung der Entwicklung der Infektion in der ersten Zeit ist unser Beitrag. Sebastian war sehr produktiv und hat Werkzeuge zur Datenerfassung aus dem Grafikfenster des TI-Nspire geschaffen.

Wolfgang Alvermann hat uns sein M-Abitur aus dem Jahr 1968 mit seiner für damals üblichen Bearbeitung geschickt. Die TI-Nspire-Ausarbeitung mit kleinen Zusätzen habe ich dann drangehängt. Es wäre doch reizvoll, wenn einige von Euch auch ihr altes Abitur (Matura in Österreich) finden und uns schicken könnten. Vielleicht gibt es da noch etwas? Eine aktuelle Abiturangabe steht bereit für den DNL#119.

Als ich Wolfgang meine Lösungen geschickt habe, hat er nur geantwortet: [„Wenn ich deine Nspire Version sehe, bin ich ja im Nachhinein noch froh, dass wir damals händisch arbeiten mussten.“](#)

Der Aufsatz von K.H. Keunecke (Tacoma Brücke) ist in deutscher Sprache in MTH118.zip enthalten.

Erfreut Euch so wie ich an den schönen Rohrflächen von Rick Nungester und seiner Fassung der entsprechenden Funktion, die sich leicht von DERIVE auf den TI-Nspire übertragen ließ.

Mit bestem Dank für Eure Zuschriften, Ideen und Mitarbeit und bis zum nächsten Mal

Euer Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & CAS-TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other *CAS* as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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### Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & CAS-TI Newsletter* will be.

Next issue:

September 2020

### **Preview: Contributions waiting to be published**

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER  
 Wonderful World of Pedal Curves, J. Böhm, AUT  
 Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT  
 Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT  
 Logos of Companies as an Inspiration for Math Teaching  
 Exciting Surfaces in the FAZ  
 BooleanPlots.mth, P. Schofield, UK  
 Old traditional examples for a CAS – What's new? J. Böhm, AUT  
 Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ  
 Tutorials for the NSpireCAS, G. Herweyers, BEL  
 Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA  
 A New Approach to Taylor Series, D. Oertel, GER  
 Statistics of Shuffling Cards, Charge in a Magnetic Field, H. Ludwig, GER  
 More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover  
 Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT  
 Affine Mappings –Treated Systematically, H. Nieder, GER  
 Investigations of Lottery Game Outcomes, W. Pröpper, GER  
 A Collection of Special Problems, W. Alvermann, GER  
 DERIVE Bugs? D. Welz, GER  
 Tweening & Morphing with TI-NspireCX-II-T, J. Böhm, AUT  
 The Gap between Poor and Rich, J. Böhm, AUT  
 Tumbling Tour in the Amusement Park, W. Alvermann, GER  
 With a Turtle to Fractals, J. Wagner, GER  
 QR-Code light, Moving a Cube, ...

### **Impressum:**

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Karl-Heinz Keunecke  
**What caused the Collapse of the Tacoma-Narrows-Bridge?**  
**or**  
**Solving Motion Problems numerically applying**  
**Newton's Fundamental Law of Motion**



Fig. 1: The carriage way of the bridge is excited to torsional vibration<sup>1</sup>

## 1. Introduction

It is one task of physics teaching to impart the importance of Newton's second axiom for calculating movement processes. This axiom, also called **Fundamental Law of Motion** says:

If a force is acting on an object then this object is accelerated in direction of the force. The acceleration is proportional to the force  $F$  and inversely proportional to the mass  $m$  of the object.

This relation is often described by  $F = m \cdot a$ , where  $a$  is the acceleration. However, from this expression one cannot recognize the importance of this law. For students  $F = m \cdot a$  is one of many other formulae which they know from physics teaching. Not before acceleration is replaced by the second derivative of the position with respect to time it becomes clear that this equation is a differential equation,

$$F(x(t)) = m \cdot \ddot{x}(t), \quad (1)$$

which allows to predict the motion of a mechanical system, if the law of force and the initial values of the motion are known.

In class the general form (1) of the fundamental law can be used only in a few special cases, because when treating mechanics in secondary II the necessary knowledge of calculus to work with differential equations is generally missed.

In order to demonstrate nevertheless the universal validity of equation (1), solutions can be calculated numerically and then graphically represented. This needs only basic knowledge of difference quotients, limits and derivatives.

<sup>1</sup> <https://sites.lsa.umich.edu/kmoore/research/tacoma-narrows-bridge/>

The following will describe a teaching course “Numerically solving differential equations”. The calculations were performed using a TI-Nspire CX CAS, which was used in class, too.

It is wanted to provoke students’ interest for the numerical solution of differential equations. So, a movie free available in the internet as motivation: “*The Collapse of the Tacoma Narrows Bridge*” is chosen.

<https://www.youtube.com/watch?v=j-zczJXSxw>  
<https://www.youtube.com/watch?v=mXTSnZgrfxM>

Then the question: “What’s the reason for this catastrophe?” can be posed.

The movie shows that the bridge vibrates heavily, and this leads finally to its collapse. Obviously, a strong wind was responsible for this disaster.

First of all, the students should investigate en forced oscillations of a spring-mass-system. By this simple model they collected experimental experiences, how and under which conditions large oscillations can result from small exciting forces. Then they could set up the respective equations of motion and finally solve them numerically.

At the end of this project the students were able – based on their results – to interpret the collapse of the Tacoma-Bridge as a resonance catastrophe.

## 2. Methods for numerically solving of Differential Equations

This paragraph describes how to solve differential equations of order 2 numerically.

Most of the dynamic processes like motion, growth and decay, any change of state etc. can be described as a mathematical model using a differential equation. As it is not possible in all cases to find an analytical solution, one has to apply a numerical approximation method. In doing so, motion is only calculated stepwise in defined time intervals. The result can be given in a table and/or as a graph.

An undamped oscillation of a spring is chosen as introductory example. The students should know this from earlier lessons. They expect as result a sinusoidal oscillation with constant amplitude. So, they can judge the quality of the numerical approximation.

Starting with initial values  $t_0$  and  $x_0$  the position coordinate  $x_1$  after time step  $\Delta t$  shall be determined.

Velocity  $v$  can be approximated by  $v \sim \frac{\Delta x}{\Delta t} v$ . The first values are  $\Delta x = x_1 - x_0$  and  $v = v_0$  which leads immediately to

$$x_1 = x_0 + v_0 \cdot \Delta t \quad (2)$$

In order to calculate the next coordinate  $x_2 = x_1 + v_1 \cdot \Delta t$  from the previous one iteratively, it needs the value of  $v_1$ . This can be achieved applying the law of force for the undamped oscillation.

There is a linear relationship between the force acting on a body of mass  $m$  and the excursion  $x$  of the oscillator during the oscillation

$$F(x) = -k \cdot x, \quad (3)$$

where  $k$  is the spring hardness.

Because of Newton’s 2<sup>nd</sup> axiom  $F(x)$  can be replaced by  $m \cdot a$ . Introducing the approximation  $a \sim \frac{\Delta v}{\Delta t}$

gives  $\Delta v = -\frac{k}{m} \cdot x \cdot \Delta t$ .

Using the initial values and equation (3) and replacing  $\Delta v$  by  $v_1 - v_0$  gives for  $v_1$ :

$$v_1 = v_0 - \frac{k}{m} \cdot x_0 \cdot \Delta t. \quad (4)$$

For calculating  $x_2$  and  $v_2$  it only needs to increase the counting variables in equations (2) and (3) by one:

$$x_2 = x_1 + v_1 \cdot \Delta t \text{ and } v_2 = v_1 - \frac{k}{m} \cdot x_1 \cdot \Delta t. \quad (5)$$

Further calculations can be done in the appropriate way.

This iteration is called Euler Method.

## 2.1 Using a Spreadsheet

The method described in the paragraph above shall be performed using the spreadsheet application of the TI-Nspire CX CAS. However, the parameters must be defined. Thus,  $\Delta t = 0.1$ ,  $x_0 = 1$ ,  $v_0 = 0$  and  $\frac{k}{m} = 1$  are set. After inserting the Lists & Spreadsheet application three columns as time, position and veloc are defined. Then the initial values are entered in the first row. Cell a2 becomes cell content of a1 increased by 0.1.

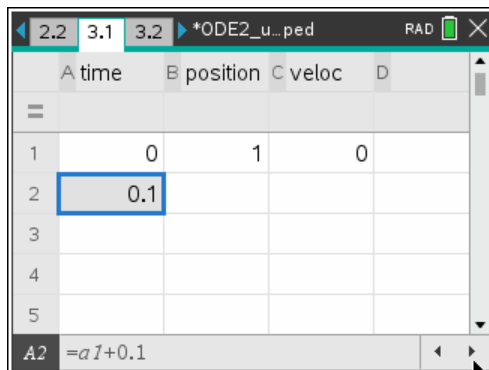


Fig. 2: Entering the initial values; increasing time by  $\Delta t = 0.1$



Fig. 3: Iterative calculation of position and velocity

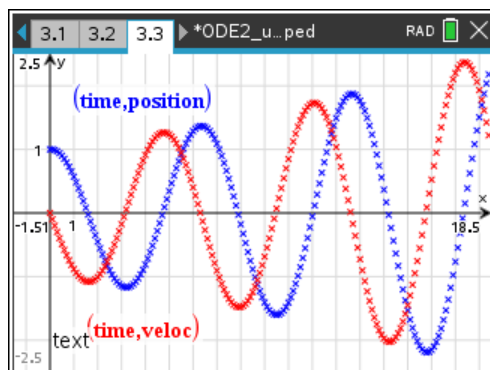


Fig. 4: Approximation of displacement and velocity of the oscillations of a spring using the Euler method

This iteration is performed for 200 rows resulting in the coordinates for position and velocity for the first 20 seconds. An upper bound is only limited by the hardware resources.

Fig. 4 shows the numerical approximation of the position of the oscillating mass (blue) and its velocity (red) dependent on time.

As expected, sinusoidal shaped curves appear. But the amplitude increases by time. This is not realistic – due to the method applied for the approximation. Fig. 5 will explain this.

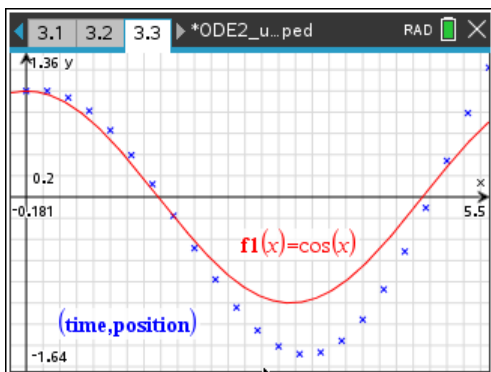


Fig. 5: Comparison of the exact solution and the approximated one

In Fig. 5 the exact solution of the differential equation (red) of an undamped oscillation  $f_1(x) = \cos(x)$  is plotted. The coordinates obtained by the Euler-method are marked by blue crosses. Compared with fig. 4  $\Delta t$  was changed to 0.25.

Calculation starts with initial values  $x_0$  and  $v_0$  which give the ordinate of the next point  $x_1 = x_0 + v_0 \cdot \Delta t$ . Since the change of the position in the first interval is  $v_0 \cdot \Delta t = 0$ ,  $x_1$  does not change:  $x_1 = x_0$ .

In the following the values at the right borders ( $x_{i+1}, v_{i+1}$ ) of the respective intervals are calculated from the values at the left borders ( $x_i, v_i$ ). This leads to discrepancies compared with the exact solution. At the beginning are the calculated ordinates greater than those of the exact solution. And this leads consequently to higher velocities, too. The next consequence is that the displacements become larger by the time, which can be observed in fig. 5.

It is obvious that the approximation goes worse when a larger time interval  $\Delta t$  is chosen. A significant reduction of the time step interval leads at one hand to an improved approximation but at the other hand to a larger amount of data. It is also possible to calculate the slope for calculation of the next point not at the begin of the respective interval but in its mid. This yields an enhanced approximation. There are numerous other methods which lead to sufficient approximations. average velocity in the intervals. A description of more methods for numerical solutions of differential equations would go too far in our teaching unit intended for secondary II students.

Paragraph 2.1 introduces thoroughly to the simple Euler-iterations by means of spreadsheets. Here the students have the chance to learn the use the iterative calculation of displacement and velocity by means of spreadsheets from scratch.

## 2.2 Application of the menu-controlled TI-Nspire CAS-tool

TI-Nspire CX CAS enables to perform the calculations presented in section 2.1 by means of a menu-driven application. That application needs simply the differential equations of the displacement and the velocity, their initial conditions and the parameters concerning the iteration. Using such a “Black Box” is acceptable, because the underlying concept has been worked out with the students in the section before).

Here again undamped oscillations shall be investigated first, in order to compare with the result shown in 2.1.

Firstly, the 1<sup>st</sup> order differential equation for the position yields:

$$\dot{x} = v. \quad (6)$$

Secondly, another 1<sup>st</sup> order differential equation for velocity  $v$  can be derived from the law of force  $m \cdot \ddot{x} = -k \cdot x$  considering that  $\ddot{x} = \dot{v}$ :

$$\dot{v} = -\frac{k}{m} \cdot x. \quad (7)$$



Equations (6) and (7) are the base for the approximation of the difference quotients, which are used like in paragraph 2.1 for calculating tables iteratively. The differential equations can be entered directly. However, the names for the occurring variables are predefined. They differ from the notation known from physics but they must be used. So, renaming is necessary. Position variable  $x$  is replaced by  $y1$  and velocity  $v$  by  $y2$ . Additionally, time  $t$  is named  $x$  and the derivative with respect to time  $t$  must be written as  $y1'$  and not as  $y\dot{1}$ . Parameter values are chosen as above:  $\Delta t = 0.1$ ,  $x_0 = 1$ ,  $v_0 = 0$  and  $\frac{k}{m} = 1$ .

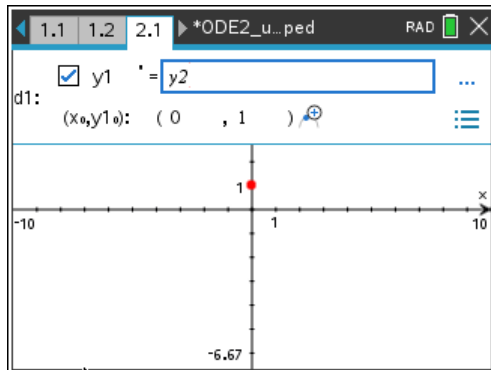


Fig. 6: Entering differential equation d1

For entering the differential equation the Graph Entry/Edit Menu in a Graphs-Application is chosen and the option Diff Eq. is selected. Now equation (5) and the initial conditions for  $y1$  can be entered using the syntax of the calculator (see fig. 6).

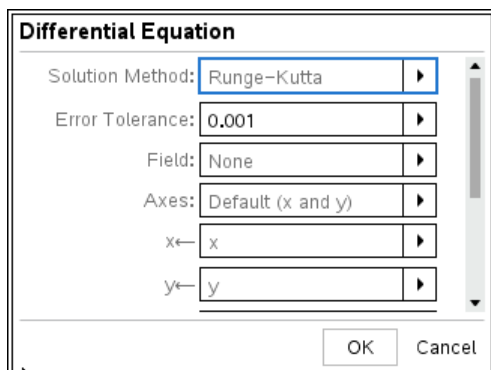


Fig. 7: Setting the parameters for the approximation

Afterwards the 2<sup>nd</sup> differential equation (6) and its initial conditions are entered (see fig. 7). The parameters for the numerical procedure must be entered as shown in fig. 7. As seen earlier that the Newton method is too inaccurate, the much more accurate Runge-Kutta method is taken. For displaying the displacement as a function of time, as axes are chosen  $x$  (time) and  $y$  (displacement). Further down in this menu the temporal interval for the numerical approximation must be defined.

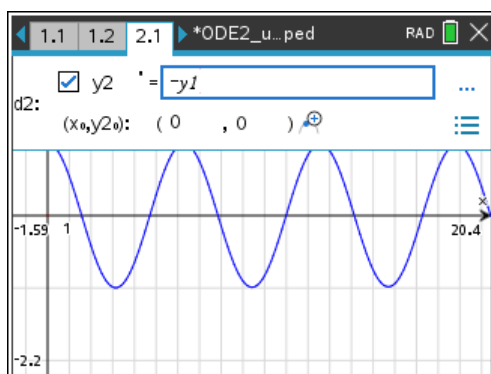


Fig. 8: Entering the DE d2 and the representation of the displacement

Fig. 8 presents the result of this approximation. In contrary to the result of the Euler method, the amplitude of this graph – again sinusoidal – seems to be constant.

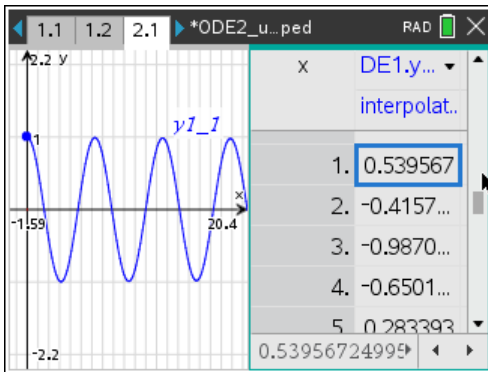


Fig. 9: Table for the positions and its graph

Fig. 9 demonstrates to students, that the menu-controlled procedure generates as well at first a list of values of time, displacement and velocity by iterations. These data are displayed then. Keypress Ctrl + T displays both, graph and list. (see fig. 9). The resulting graphs in fig. 8 and fig.10 can be modeled by trigonometric functions. When for instance  $\cos x$  is drawn in the same graph, both graphs coincide exactly

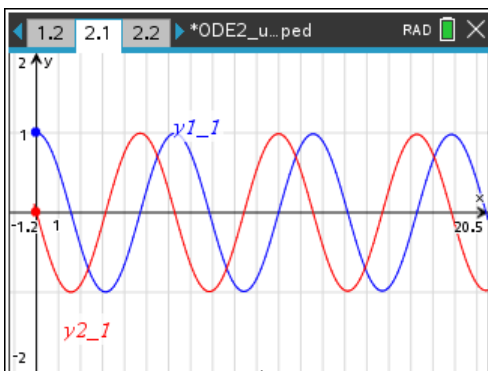
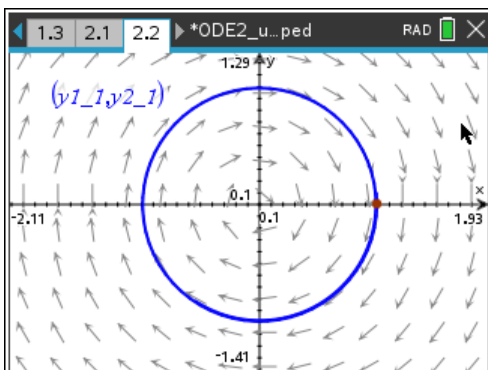


Fig. 10: Displacement and velocity

Both the displacement  $y_1$  and the velocity  $y_2$  have been calculated as it is seen in fig. 10. Due to the initial conditions and the assumption  $\frac{k}{m} = 1$ , what means  $\omega = 1 \frac{rad}{s}$ , two phase-shifted sine waves with equal amplitudes appear.

Phase relation of both graphs can be recognized in a phase diagram (fig. 11), where  $y_2$  is drawn as function of  $y_1$ . The phase diagram gives a circle only when the amplitudes of both waves are equal and the phase shift between both is  $\frac{\pi}{2}$ .

Fig. 11: Phase diagram  $y_2$  vs  $y_1$ 

Students can reproduce such diagrams, when they draw two sine functions against each other and vary the phase shift between 0 and  $2\pi$ .

This diagram shows the direction vectors of the motion in several points. We can recognize that the diagram is passed through clockwise starting at point (0|1).

### 3. Examples from Physics Lessons

Paragraphs 2.1 and 2.2 served as introduction into computer assisted numerical solution of differential equations. This procedure is an important tool in physics. As it has demonstrated before, this tool can also easily be used in physics lessons. The procedures enable to answer questions, which cannot be treated without computer assistance. Among others there exist physical problems without analytic solutions in closed form or problems which cannot be solved with the mathematical knowledge of secondary II level.

Students should proceed as follows:

- At first, they develop the law of force of a motion and deduce then the respective 2<sup>nd</sup> order differential equation.
- Thereafter they replace this differential equation by two 1<sup>st</sup> order differential equations for position and velocity which enable a numerical solution.
- They enter the differential equations into the CAS-calculator and obtain the graphic representation of position and velocity as function of time.
- They have also access to the calculated values given in a table.

Two examples shall be demonstrated in the next section.

#### 3.1 Pendulum Swinging

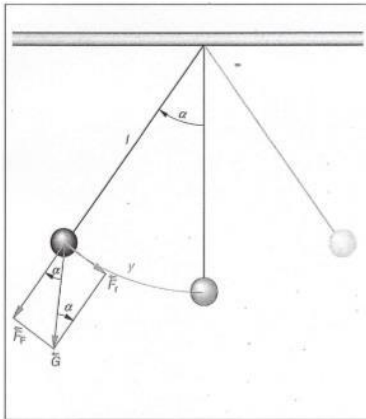


Fig. 12: Forces acting on a rod pendulum

Gravity pendulums generally don't execute undamped (harmonic) oscillations. Only for small swings the motion can be considered approximatively as an undamped oscillation.

In order to work out the acting forces, an idealized model is assumed, the so-called mathematical pendulum. It consists of a point-shaped mass fixed on a massless rod moving without friction on a pivot (fig. 12).

For a rotation around the pivot point after being displaced by the angle  $\alpha(t)$  repulsive forces  $\vec{F}_r$  are acting which are directed tangent to the circular arc. This force can be calculated according fig. 12 and because the gravity force  $\vec{G}$  can be substituted by  $m \cdot g$ , it results in

$$F_r(\alpha(t)) = -m \cdot g \cdot \sin(\alpha(t)), \quad (8)$$

where  $m$  is the mass of the body and  $g$  the acceleration of gravity. In the following  $\alpha$  is always  $\alpha(t)$ .

From equation (8) Thus, it can be seen that there is no proportional relation between repulsive force and the displaced angle. Hence, the swing of the pendulum is not harmonic oscillation.

The respective differential equation can be derived from the above law of force. Due to Newton's 2<sup>nd</sup> axiom the left side of (8) is replaced by  $m \cdot a_t$ . The acceleration  $a_t$  of a body moving along a circle with the radius  $l$  is  $l \cdot \ddot{\alpha}$ . Division (8) by  $l \cdot m$  results in

$$\ddot{\alpha} = -\frac{g}{l} \sin(\alpha). \quad (9)$$

By introduction of an additional variable – angle velocity  $\omega = \dot{\alpha}$  – the system of two 1<sup>st</sup> order differential equations can be deduced from the 2<sup>nd</sup> order differential equation:

$$\dot{\alpha} = \omega \text{ and } \dot{\omega} = -\frac{g}{l} \sin(\alpha). \quad (10)$$

Using the syntax of the calculator this must be entered as:

$$y1' = y2 \text{ and } y2' = -\frac{g}{l} \sin(y1). \quad (11)$$

For convenience  $\frac{g}{l} = 1 \cdot \frac{\text{rad}^2}{\text{s}^2}$  is chosen and the initial velocity is set to  $0 \cdot \frac{\text{m}}{\text{s}}$ .

In order to provide variable initial conditions for  $y1$ , a slider is used to vary the values of  $y1(0)$  from the interval  $[0; \pi]$ .

System (11) is entered in the calculator. Both displacement (black) and angle velocity (red) as well are displayed dependent on time.

The following figures 13 and 14 give a first impression of the run of displacement and velocity dependent on the respective initial value of the displacement. The scales on the axes remain unchanged in order that changes in the magnitude of displacement, velocity and frequency become clear.

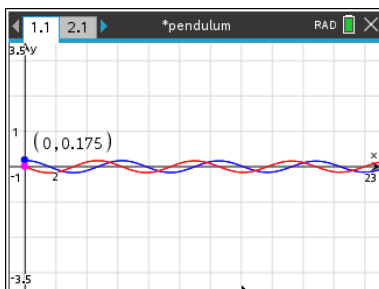
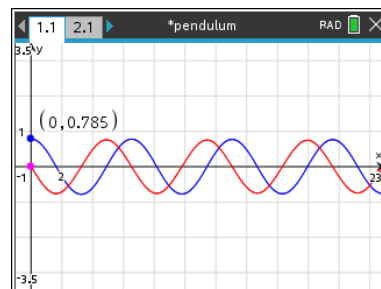
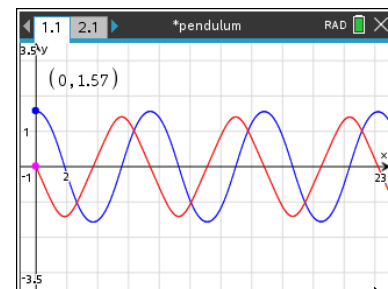


Fig. 13:  $10^\circ \sim 0.175 \text{ rad}$



$45^\circ \sim 0.785 \text{ rad}$



$90^\circ \sim 1.57 \text{ rad}$

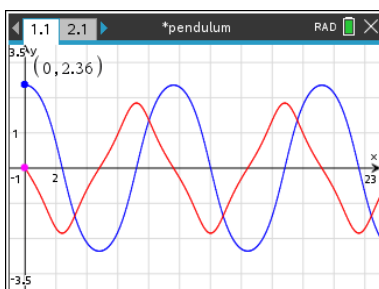
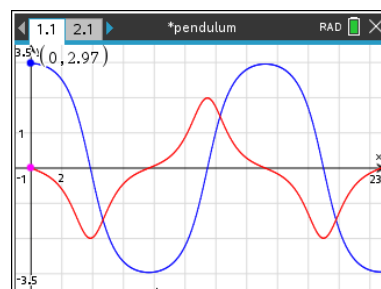
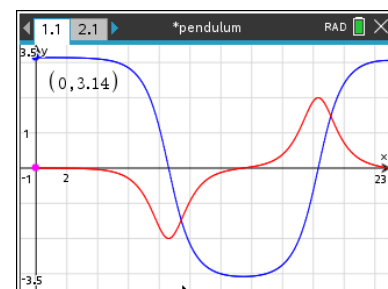


Fig. 14:  $135^\circ \sim 2.36 \text{ rad}$



$170^\circ \sim 2.79 \text{ rad}$



$179.90^\circ \sim 3.14 \text{ rad}$

As long as displacement at the beginning is small (see 1<sup>st</sup> picture in fig. 13), the graphs of  $y1$  and  $y2$  are sinusoidal. In this range the motion shows all indications of a harmonic oscillation. If the displacement increases, the deviation from harmonic oscillation becomes more and more significant. Moreover, the frequency of the oscillation changes with the magnitude of the initial displacement of the pendulum.

The acting forces, causing the pendulum motion are described by equation (8). In contrary to the linear motion law (3) the forces are restricted by the factor  $\sin(\alpha)$ . This enforces a slowdown of the motion and an enlargement of the oscillation time. Because in particular the repulsive forces at angles above  $90^\circ$  decrease – instead of increasing like in undamped oscillations – there are significant deviations from sinusoidal graphs.

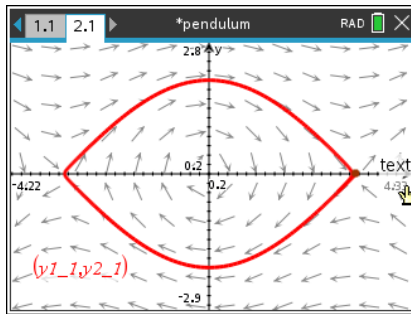


Fig. 15: Phase diagram;  
initial value  $179.9^\circ \sim 3.14$  rad

Instead of investigating the temporal behavior of  $y_1$  and  $y_2$  it is also possible to look at the phase diagram.

Fig. 15 shows the graph of  $y_2$  dependent on  $y_1$  for an initial displacement of  $179.9^\circ$  (fig. 14, last screen). In the presentation motion starts at the red point (maximum displacement). The direction arrows indicate that the curve is run through clockwise. Comparison with fig. 11 shows clearly the difference to a harmonic (undamped) oscillation).

### 3.2 Damped and Enforced Oscillations

The following investigation shall help to understand the process of the Tacoma-Bridge collapse. Obviously, the bridge executes vibrations under influence of the wind. It is proposed to investigate enforced oscillations on a simple model at first. It makes sense to watch and study motions in reality before mathematical modelling the procedures.

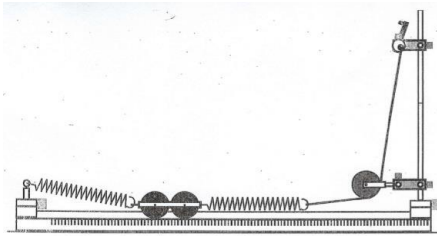


Fig. 16: Experiment to investigate damped and enforced oscillations

As a simple experiment a cart on a track<sup>2</sup> has been c which is fixed between two springs. When periodical caused by rotation of the eccentric wheel are acting cart then it performs enforced oscillations. After own iments the students will find out:

1. After response time the cart moves in the rhythm of the acting force.
2. For a relatively small frequency range of the excitation the displacement of the cart exceeds significantly the displacement of the excitation.

#### Damped Oscillations

For simulating the experiment knowledge of the forces acting on the cart is necessary. At first, the repulsive force of the springs – given in equation (2) –  $f(x) = -k \cdot x$  acts on the cart.

It is known that the amplitude of real oscillations decreases over time. So, besides the repulsive force another force, the friction, acts in the same direction, which is in general proportional to the velocity  $v$  of the oscillating body:

$$F_f = -r \cdot v = -r \cdot \dot{x}$$

where  $r$  is a measure for the strength of damping.

Thus, the law for the damped oscillation of the cart results in:

$$F(x(t)) = -k \cdot x(t) - r \cdot \dot{x}(t) = m \cdot \ddot{x}(t). \quad (12)$$

In analogy to the procedure presented in paragraph 2.2 one obtains the following system of 1<sup>st</sup> order differential equations:

$$\dot{x}(t) = v(t) \text{ and } \dot{v}(t) = -\frac{k}{m} \cdot x(t) - \frac{r}{m} \cdot \dot{x}(t),$$

<sup>2</sup> 2013, MEKRUPHY, Mechanik 6, Erzwungene Schwingungen, p 24

which is written in the syntax of the calculator as follows:

$$y1' = y2 \text{ and } y2' = -\frac{k}{m} \cdot y1 - \frac{r}{m} \cdot y2. \quad (13)$$

For calculating a special solution values of the constants  $m$ ,  $k$  and  $r$  are needed. The students know from physics lessons that  $\frac{k}{m} = \omega_0^2$  is the resonance frequency. Oscillation time of the eigenfrequency (fig. 16) can be measured and the angular velocity  $\omega_0$  can be determined from it. Alternatively,  $\omega_0$  can be determined from the spring constant  $k$  and the mass  $m$  of the cart.

$$\frac{k}{m} = 6.7 \frac{\text{rad}^2}{\text{s}^2}.$$

Parameter  $\frac{r}{m}$ , which describes the damping cannot be taken from the experimental set-up in such an easy way. Therefore, a slider  $b$  is introduced for  $\frac{r}{m}$ . It was set such that the time flow of the numerically calculated oscillation matches with the time flow of the displacement of the cart (fig. 16) in the best possible way. Comparison lead to  $b = 0.25$ . It will be defined later on.

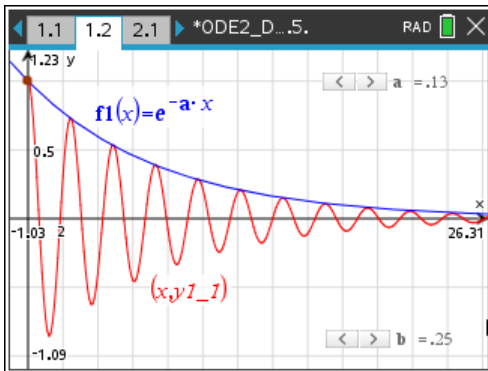


Fig. 17: Damped oscillation and exponential function

Taking the given parameters both differential equations are entered in respective menu in the Graphs-application. The result is displayed in fig. 17.

The calculated damped oscillation is described by a cosine-function with obviously exponential decreasing amplitudes. In order to confirm this, the graph of  $f1$  was additionally plotted. The slider for variable  $a$  was adjusted so, that the graph of  $f1$  touches and enclosed the relative maxima of the graph of the numerical solution. This is the case for  $a \sim 0.13 \text{ s}^{-1}$ .

From these observations can be concluded that the time flow of the oscillation is given by:

$$e^{-a \cdot x} \cdot \cos(\sqrt{6.7} \text{s}^{-1} \cdot x). \quad (14)$$

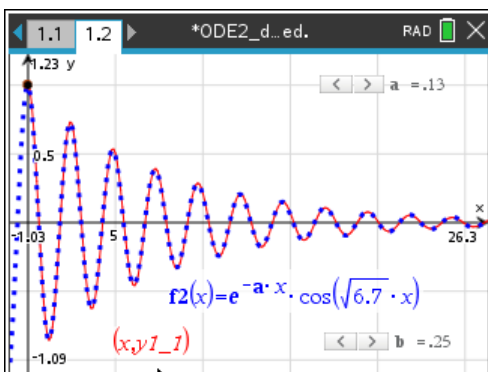


Fig. 18: Modelling the graph by function  $f2$

In fig. 18 the numerical solution (red) and the graph according to expression (14) (blue and dotted) is modelled. The algebraic model (equation 14) and the numerical approximation match excellently for  $a = 0.13 \text{ s}^{-1}$ .

The numerical solution can be described algebraically – in the notation used in physics – by the following functional equation

$$x(t) = e^{-a \cdot t} \cdot \cos(\sqrt{6.7} \text{s}^{-1} \cdot t). \quad (15)$$

Yet the relationship between the parameters  $a$  and  $b$ , which both describe the strength of damping, is to find out. The example above shows that the value of  $b$  is approximately twice the value of  $a$ . This relation is valid for any other parameter  $a$  and  $b$ . Using the exact solution of the differential equation one can see that the decay of the oscillations is determined by the factor  $e^{-\frac{1}{2} \frac{r}{m} t}$ . With students one can only

suppose that  $a = \frac{1}{2} \cdot b = \frac{1}{2} \cdot \frac{r}{m}$ . With this assumption one can model the numerical solution by a function.

$$x(t) = e^{-\frac{1}{2} \frac{r}{m} t} \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t\right). \quad (16)$$

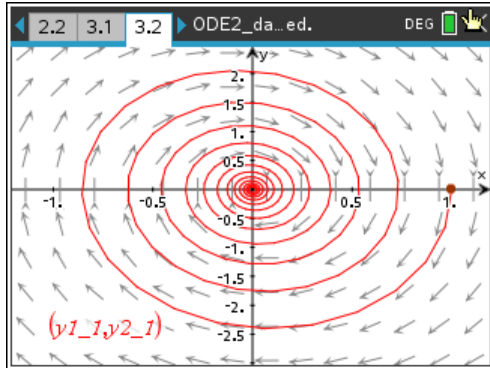


Fig. 19: Phase diagram of a damped oscillation

Finally, the phase diagram of the damped oscillation is displayed (fig. 19), which presents the velocity of the swinging cart as function of its displacement. It shows clearly the exponential decrease of the oscillations.

### Enforced Oscillations

After the detailed investigations of damped oscillations of the cart the differential equation for the motion of the cart (fig. 16) under influence of a periodical external force can be set up.

The external force for a continuous rotation of the crank is  $F_0 \sin(\omega t)$ , where  $\omega$  the frequency of the exciting force and  $F_0$  the amplitude of the force. Then the occurrent forces are given by:

$$m \cdot \ddot{x} = -k \cdot x(t) - r \cdot \dot{x}(t) + F_0 \cdot \sin(\omega t). \quad (17)$$

The respective inhomogeneous 2<sup>nd</sup> order differential equation result in:

$$\ddot{x} = -\frac{k}{m} \cdot x(t) - \frac{r}{m} \cdot \dot{x}(t) + \frac{F_0}{m} \cdot \sin(\omega t). \quad (18)$$

For the numerical solution of equation (18) again a system of two 1<sup>st</sup> order differential equations – for the position and the velocity – is needed. Using the syntax of the calculator one obtains:

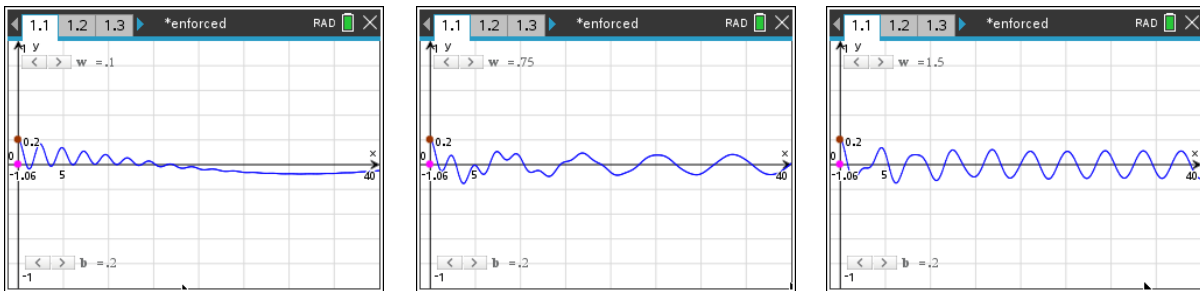
$$y1' = y2 \text{ and } y2' = -\frac{k}{m} \cdot y1 - \frac{r}{m} \cdot y2 + \frac{F_0}{m} \cdot \sin(\omega x). \quad (19)$$

Following parameters are taken:

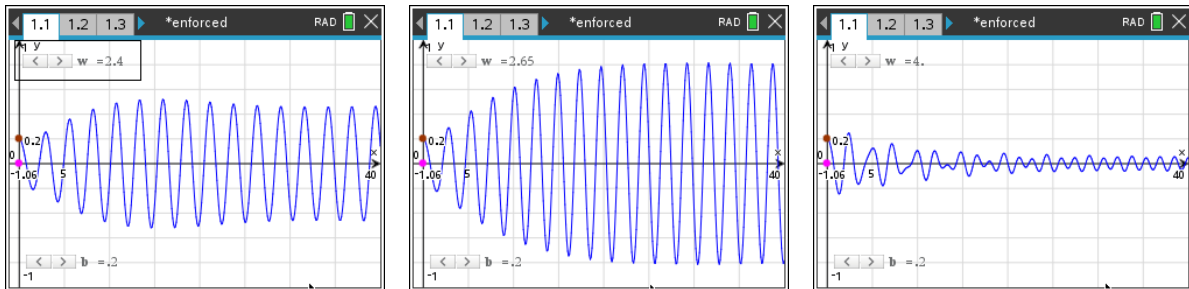
$\frac{k}{m} = 6.7 \frac{\text{rad}^2}{\text{s}^2}$ ,  $\frac{r}{m} = 0.2 \cdot \frac{1}{\text{s}}$ , angular frequency  $0.1 \cdot \frac{\text{rad}}{\text{s}} \leq \omega \leq 4 \cdot \frac{\text{rad}}{\text{s}}$ . The time interval is set on 40 s.

the amplitude of the acceleration forced by the hank is given by  $\frac{F_0}{m} = 0.5 \frac{1}{\text{s}^2}$ . The initial values then are  $y1(0) = 0.2$ ,  $y2(0) = 0$ .

In the following presentations the angular frequency is varied using a slider (variable  $w$ ). Scaling of the axes is kept constant so that the differences in size of the amplitudes and frequencies are recognizable.

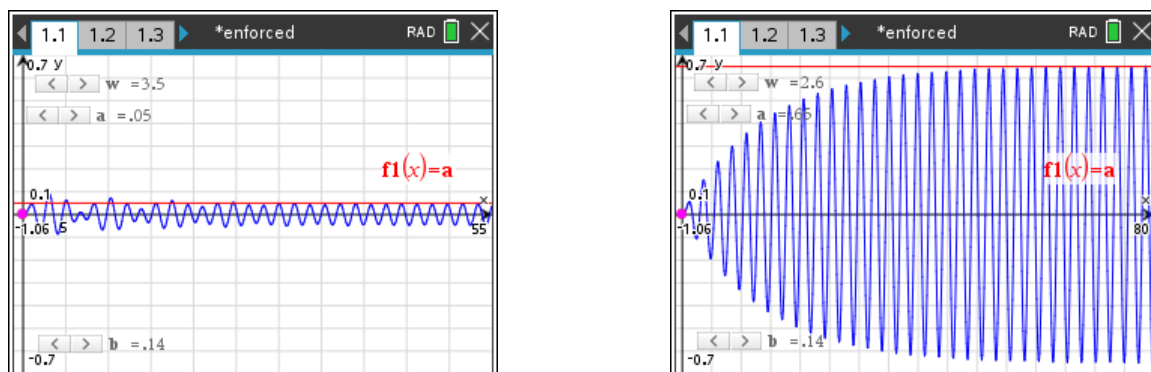
Fig. 20: Period  $T \cong 63s$  $T \cong 8.4s$  $T \cong 4.2s$ 

At the beginning of the excitation of the oscillation the displacement of the cart is displaced by the motion of the crank. As a result the cart is excited to damped oscillation with the natural frequency of the system. These oscillations are superimposed with the periodical excitation. This can be seen in fig. 20. After a transient phase of approximately 20s the damped oscillation is decayed and after that follows a stationary oscillation of the cart with the frequency of the excitation. For very small frequencies the amplitude of the cart is just as great as the displacement by the eccentric.

Fig. 21: Period  $T \cong 2.62s$  $T \cong 2.46s$  $T \cong 1.57s$ 

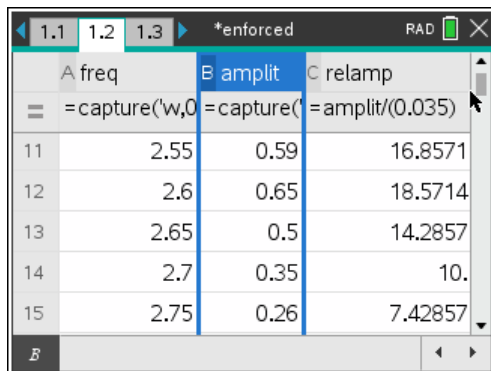
As soon as the excitation frequency approaches the resonance frequency of the clamped cart (fig. 21, pictures 1 and 2) the amplitude of the motion of the cart grows up to nearly the twentyfold of the excitation displacement. After this state of resonance, the oscillation decreases again and its amplitude tends towards zero for high frequencies.

This qualitative evaluation of the numerical solution offers the students a physical explanation of the vibrations of the bridge and its collapse. From the point of physics, it is more satisfying if a quantitative statement for enforced oscillations is also possible. From the graphic representations of the solutions for various frequencies of excitation one can determine the amplitude of the stationary displacements of the cart after transient effects have been disappeared.

Fig. 22: Plotting  $f1(x)$  passing the maximum values of the displacements



In graphical displays of the application *Diff Eq* one cannot display the coordinates of points of graphs. In order to find the coordinates of the maxima during the steady state, the graph to  $f1(x) = a$  is plotted. Parameter  $a$  is ruled by a slider. The graph of  $f1$  is parallel to the  $x$ -axis. The value of  $a$  can adjust so, that the line touches and enclose the displayed maxima as shown in fig 22.



A freq	B amplit	C relamp
=capture("w,0	=capture(	=amplit/(0.035)
11	2.55	0.59
12	2.6	0.65
13	2.65	0.5
14	2.7	0.35
15	2.75	0.26

Fig. 23: Frequencies and amplitudes

The frequencies and the respective amplitudes are measured and transferred to the TI-Nspire spreadsheet application. The measured values are divided by the value of the amplitude of the periodic excitation of the eccentric and displayed then in the third column. (This standardization makes the comparison of different systems possible.).

Now the relative amplitudes can be plotted versus the frequencies (fig. 24). The graph shows very clear the resonance magnification when excitation acts with resonance frequency. In this case the amplitude of oscillation of the cart is almost twenty times greater than the maximum displacement of the excitation.

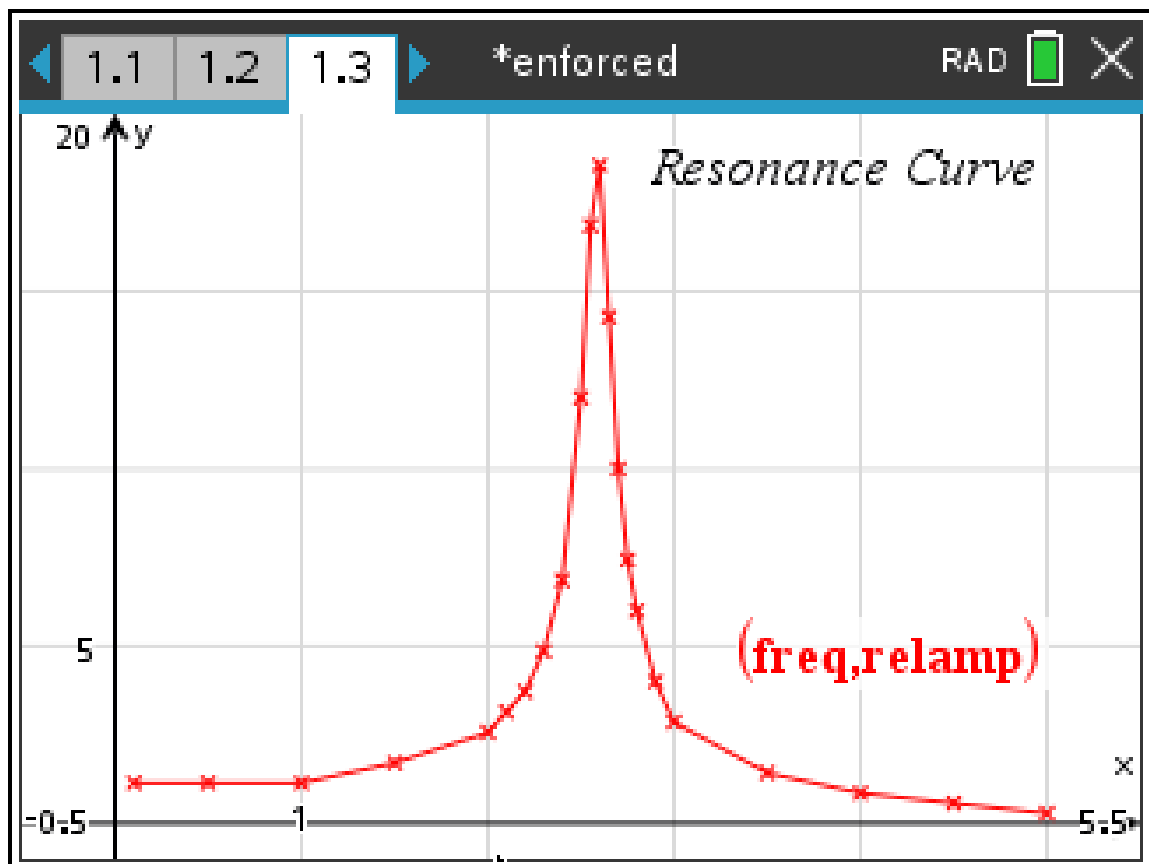


Fig. 24: Resonance curve of the model for enforced oscillation

Calculations can be continued for other values of damping, to understand the phenomenon of enforced oscillations under other conditions, too.

With this investigated model for enforced oscillations it was not possible to increase the energy input so far, that a resonance catastrophe occurs. As a little candy for holding on so long one should as a teacher let an empty wine glass split using a loudspeaker with the appropriate resonance frequency and enough sound volume. Finally, the **definition of the concept “resonance catastrophe”** found by Wikipedia is presented to the students and discussed with them:

In mechanics and construction, a **resonance disaster** describes the destruction of a building or a technical mechanism by induced vibrations at a system's resonance frequency, which causes it to oscillate. Periodic excitation optimally transfers to the system the energy of the vibration and stores it there. Because of this repeated storage and additional energy input the system swings ever more strongly, until its load limit is exceeded.

#### 4. Conclusion

The functional description of the resonance curve can only be deduced from the exact solution of the corresponding differential equation. Students lack of the considerable skills in mathematics for such calculation.

This article on numerical solutions of differential equations has shown, that students have the ability to determine the resonance curve point by point (fig. 24) This result demonstrates, that the numerical solving of differential equations could be very helpful in physics lessons.

## Rüdeger Baumann

Wolfgang Ludwicki informed me end of March that our member for many years, Rüdeger Baumann, passed away 10 July 2019. Rüdeger was a pioneer in didactics of teaching information technology and combining it with mathematics education from the very beginning. He was author of numerous books and contributions in journals (LOG IN a.o.). He could finish his last work “Object Oriented Modelling with SAMLLTALK/Squeak” in the LOG In journal.



I remember his passionate presentations at the Vienna Conference VISIT-ME in 2002<sup>[\*]</sup>. Although he was an expert in computer assisted math teaching, he preferred sending his contributions in “old-fashioned” letters.

Rüdeger contributed many articles for our newsletter (13). His contributions and his – sometimes controversial opinions - are his legacy for the DUG-Community.

You can find an obituary and appreciation of his work at <https://www.log-in-verlag.de/digitale-bildung/>

[\*] His lecture given at VISIT-ME 2002 is part of MTH118.zip.

## Tacoma-Collapse treated with DERIVE

The Euler-method for numerically solving DEs is not implemented in DERIVE - so Help Yourself:

Euler Method:

```

ode2(x0, v0, dt, n, t0, xn, vn, spr, i, rn) :=
  Prog
  t0 := 0
  spr := [[t0, x0, v0]]
  i := 1
  Loop
  If i > n
    RETURN spr
  t0 := t0 + dt
  vn := v0 - x0*dt
  xn := x0 + v0*dt
  spr := APPEND(spr, [[t0, xn, vn]])
  x0 := xn
  v0 := vn
  i := i + 1

```

#2: ode2(1, 0, 0.1, 10)

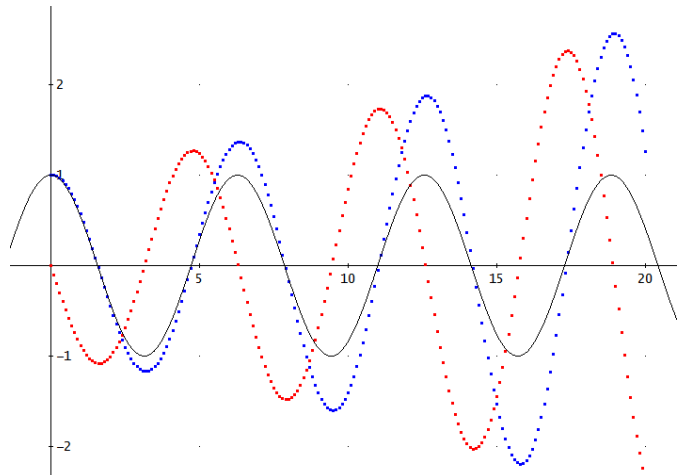
#4: (ode2(1, 0, 0.1, 200)) COL [1, 2]

#5: (ode2(1, 0, 0.1, 200)) COL [1, 3]

#6: COS(x)

	0	1	0
	0.1	1	-0.1
	0.2	0.99	-0.2
	0.3	0.97	-0.299
	0.4	0.9401	-0.396
#3:	0.5	0.9005	-0.49001
	0.6	0.851499	-0.58006
	0.7	0.793493	-0.6652098999
	0.8	0.7269720099	-0.7445591999
	0.9	0.6525160899	-0.8172564009
	1	0.5707904499	-0.8825080099

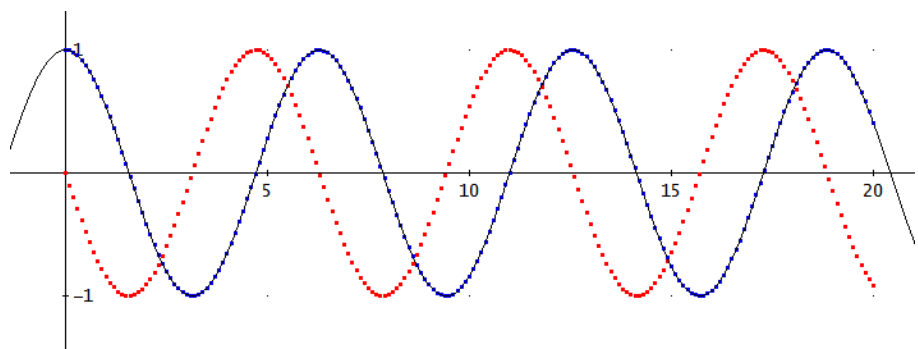
Fig. 4 from above



#8: (RK([y2, -y1], [t, y1, y2], [0, 1, 0], 0.1, 200)) COL [1, 2]

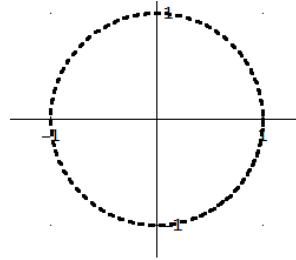
#9: (RK([y2, -y1], [t, y1, y2], [0, 1, 0], 0.1, 200)) COL [1, 3]

Fig. 10 from above



#10: (RK([y2, -y1], [t, y1, y2], [0, 1, 0], 0.1, 200)) COL [2, 3]

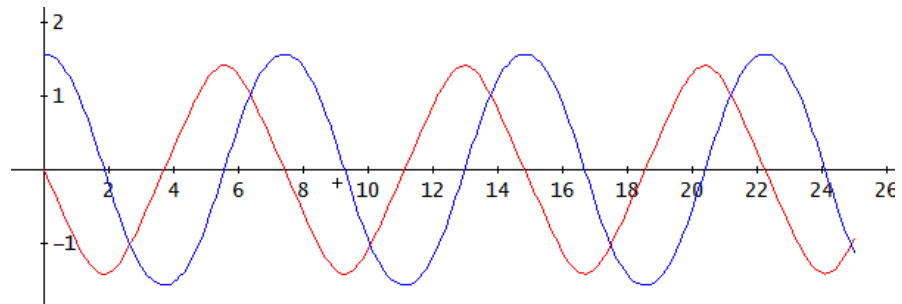
Fig. 11 from above



#13: (RK([y2, -SIN(y1)], [t, y1, y2], [0, 1.57, 0], 0.1, 250)) COL [1, 2]

#14: (RK([y2, -SIN(y1)], [t, y1, y2], [0, 1.57, 0], 0.1, 250)) COL [1, 3]

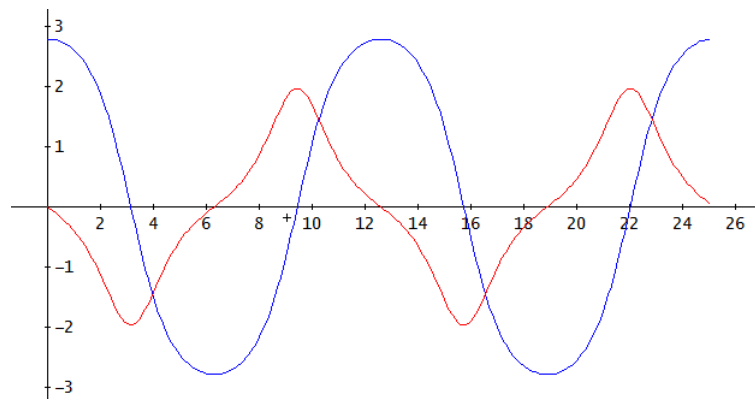
Fig. 13-3 from above



#15: (RK([y2, -SIN(y1)], [t, y1, y2], [0, 2.79, 0], 0.1, 250)) COL [1, 2]

#16: (RK([y2, -SIN(y1)], [t, y1, y2], [0, 2.79, 0], 0.1, 250)) COL [1, 3]

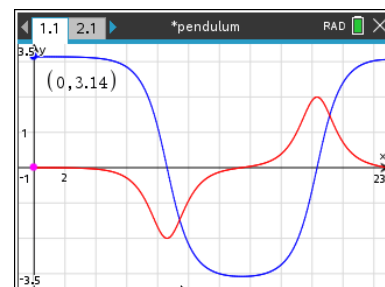
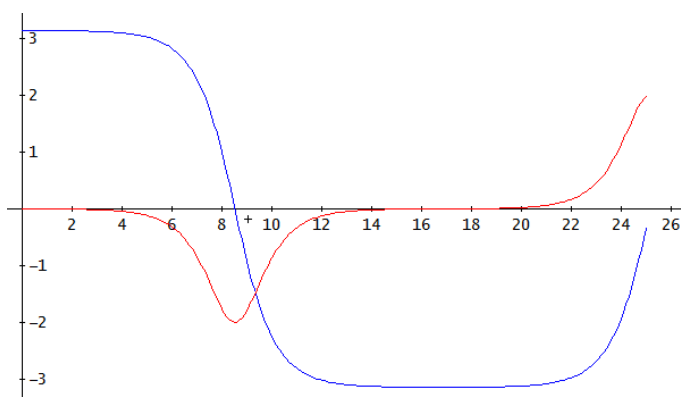
Fig. 14 from above



#17: (RK([y2, -SIN(y1)], [t, y1, y2], [0, 3.14, 0], 0.1, 250)) COL [1, 2]

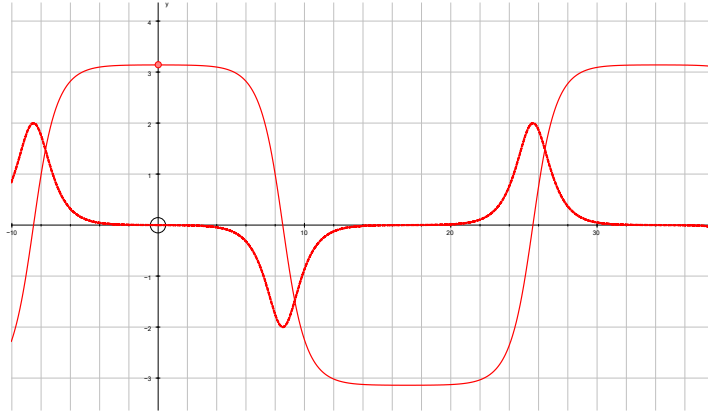
#18: (RK([y2, -SIN(y1)], [t, y1, y2], [0, 3.14, 0], 0.1, 250)) COL [1, 3]

Fig. 14-3 from above



Comparing the graphs one can notice a significant difference in the period length. What is correct? TI-Nspire or DERIVE? I asked Autograph (see below) and then *MATHEMATICA*.

### Autograph



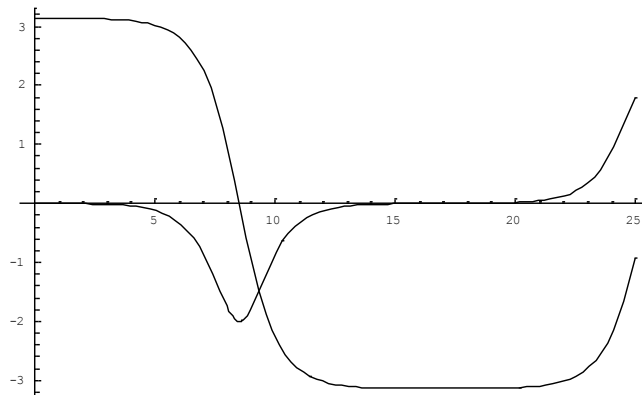
### *MATHEMATICA*

```
NDSolve[{y1'[x]==y2[x],y2'[x]==-Sin
[y1[x]],y1[0]==3.14,y2[0]==0},{y1,y2},{x,0,25}]
```

```
{{y1->InterpolatingFunction[{{0.,25.}},<>],y2->InterpolatingFunction[{{0.,25.}},<>]}}
```

-Graphics-

```
Plot[{Evaluate[y2[x]/.%10],Evaluate[y1[x]/.%10]},{x,0,25}]
```

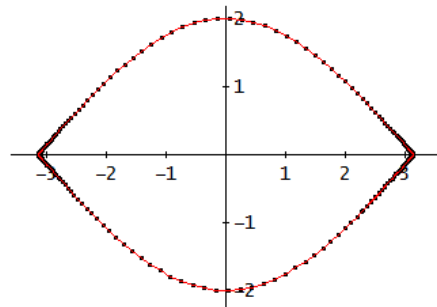


-Graphics-

It seems to be that close to  $\pi$  TI-NspireCAS gives a wrong approximation.

```
#21: (RK([y2, - SIN(y1)], [t, y1, y2], [0, 3.14, 0], 0.1, 400)) COL [2, 3]
```

Fig. 15 from above

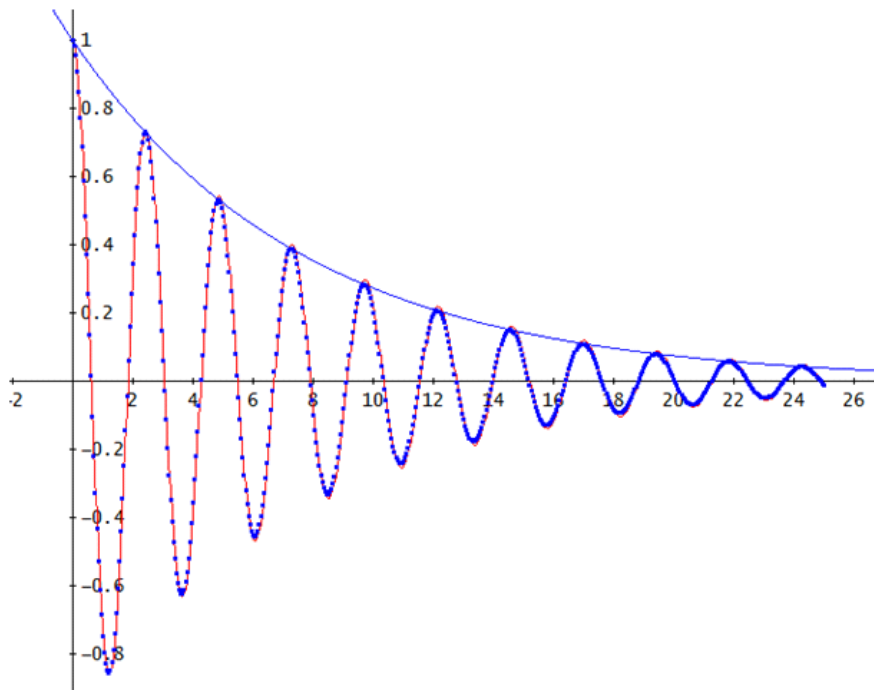


#22: (RK([y2, - 6.7·y1 - 0.25·y2], [t, y1, y2], [0, 1, 0], 0.05, 500)) COL [1, 2]

#23:  $e^{-0.13 \cdot x}$

#24: TABLE( $e^{-0.13 \cdot x} \cdot \cos(\sqrt{6.7 \cdot x})$ , x, 0, 25, 0.05)

Fig. 17 & 18



Compare with the

exact solution:

#25: DSOLVE2\_IV(0.25, 6.7, 0, x, 0, 1, 0)

$$\#26: e^{-x/8} \left( \cos\left(\frac{\sqrt{10695 \cdot x}}{40}\right) + \frac{\sqrt{10695} \cdot \sin\left(\frac{\sqrt{10695 \cdot x}}{40}\right)}{2139} \right)$$

#27:  $e^{-0.125 \cdot x} \cdot (\cos(2.585415827 \cdot x) + 0.04834812206 \cdot \sin(2.585415827 \cdot x))$

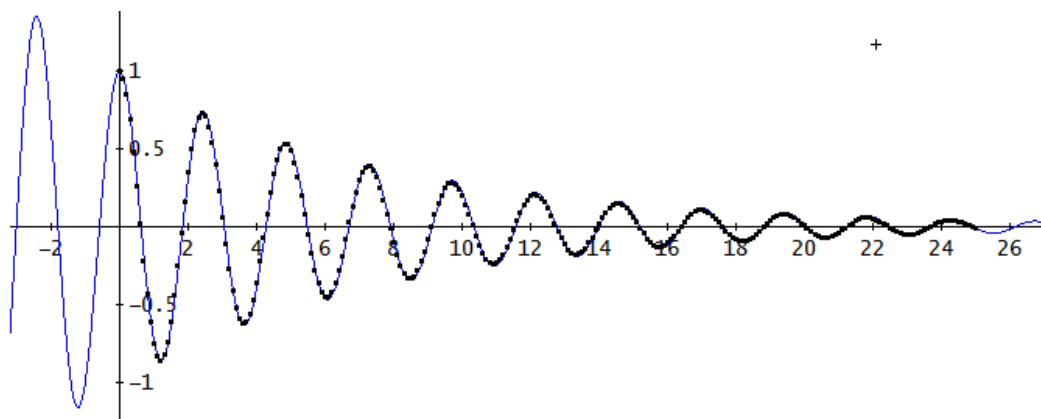
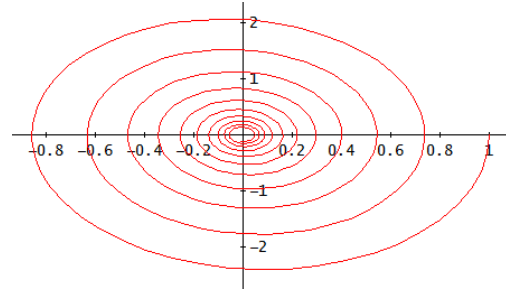


Figure 19



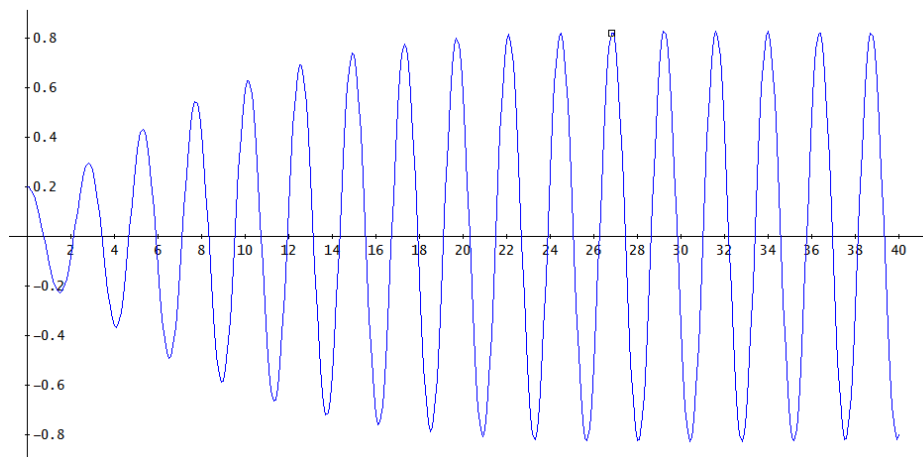
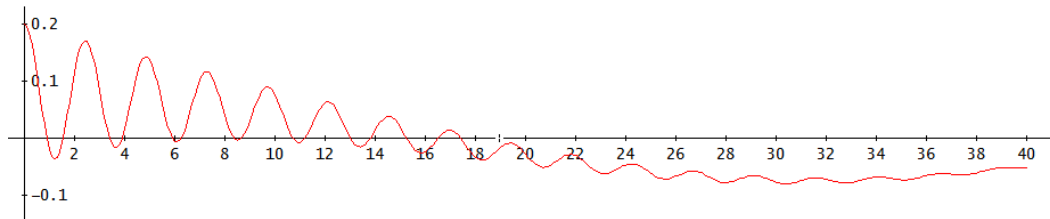
We cannot vary the angular frequency using a slider, so it is necessary to plot the single curves and find the maximum amplitude, which can be done using the trace-tool.

Figures 20-1, 21-2 and 22-2:

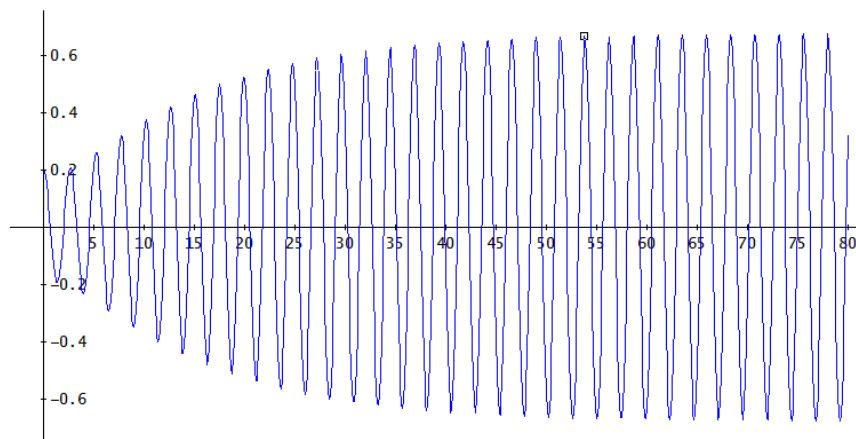
#28: (RK([y2, - 6.7·y1 - 0.2·y2 + 0.5·COS(0.1·t)], [t, y1, y2], [0, 0.2, 0], 0.1, 400)) COL [1, 2]

#29: (RK([y2, - 6.7·y1 - 0.2·y2 + 0.5·COS(2.65·t)], [t, y1, y2], [0, 0.2, 0], 0.1, 400)) COL [1, 2]

#30: (RK([y2, - 6.7·y1 - 0.14·y2 + 0.25·COS(2.6·t)], [t, y1, y2], [0, 0.2, 0], 0.1, 800)) COL [1, 2]



Maximum ~ 0.82

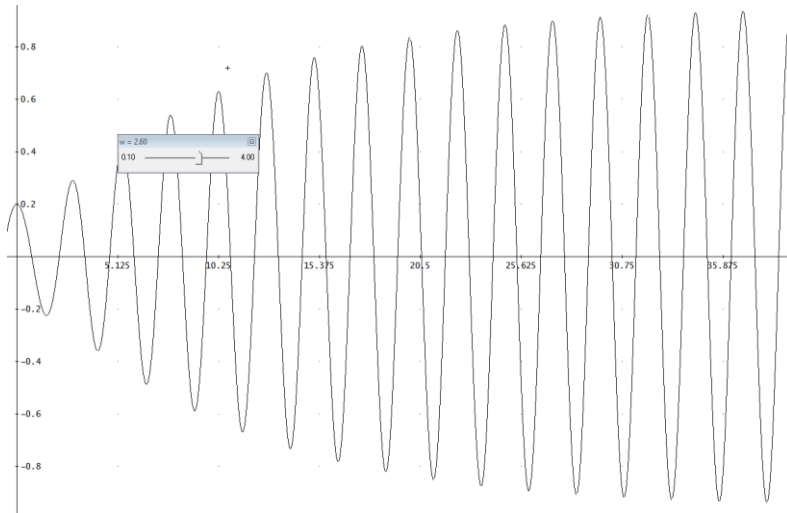


Maximum ~ 0.66

Introducing a slider for  $w$  is possible when DSOLVE2-function is applied:

Sliderbar for  $0.1 \leq w \leq 4$

DSOLVE2\_IV(0.2, 6.7, 0.5·COS(w·x), x, 0, 0.2, 0)



Reproducing Karl-Heinz' project it was too boring for me to plot each RK-approximation and finding its maximum amplitude. So, I remembered my favorite DERIVE-function: VECTOR:

```
MAX((RK([y2, - 6.7·y1 - 0.14·y2 + 0.25·COS(2.6·t)], [t, y1, y2], [0, 0.2, 0], 0.2, 400)) COL [2]' )
```

#32: 0.6693017476

#31 returns the maximum amplitude for  $w = 2.6$  without plotting. Wrap this function into a VECTOR-construction for  $2 \leq w \leq 3.5$  and collect the points  $[w, \text{resp. Max}]$ :

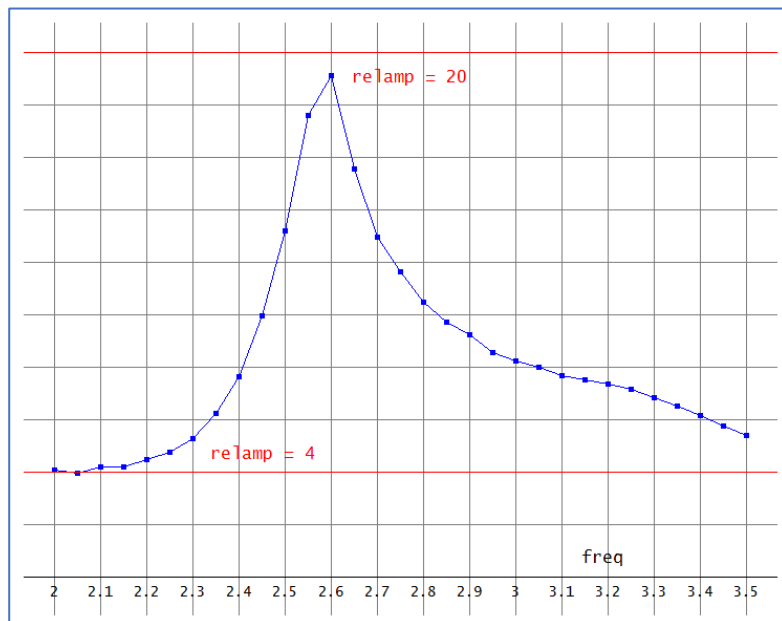
```
VECTOR(
  [
    w,
    MAX((RK([y2, - 6.7·y1 - 0.14·y2 + 0.25·COS(w·t)], [t, y1, y2], [0, 0.2, 0], 0.2, 400)) COL [2]' )
  ], w, 2, 3.5, 0.05)

```

[y = 4, y = 20, y = 20]

Plotting the expression (without evaluating) we get in one step the resonance curve.

Karl-Heinz was not very happy with this curve, because it looks a bit different. I didn't like to cheat, this is the DERIVE-product. (RK-based)





## 3D Lissajous Tubes

by Rick Nungester, 4/23/20

I am a retired electrical/software engineer and Derive enthusiast since 1993. In 2002 I became interested in 3D Lissajous parametric curves, for example  $x = \sin(3t)$ ,  $y = \sin(4t)$ ,  $z = \cos(5t)$ ,  $-\pi \leq t \leq \pi$ . Using COS for the z equation instead of SIN avoids the curve intersecting itself. To turn the curves into tubes I found GRAPHICS.MTH function SPACE\_TUBE but didn't like that its tubes appear to be non-circular as they go around tight corners. Function CURVE\_TO\_TUBE improves this problem by keeping the tube circle perpendicular to the curve. Details are in file 3DLissajous.dfw. The images look particularly nice using "Rotate Plots" in Derive as demonstrated in video file 3DLissajous.mp4.

(Editor: Rick sent his great 102MB-file. I cannot add it to the files in MTH118.zip.)

#1: Lissajous 3D Space Tubes by Rick Nungester Dec 27, 2002. Last modified Apr 29, 2020.

#2: CURVE\_TO\_TUBE() moves a circle of radius r along the space curve  $v = [x(t), y(t), z(t)]$ , in a plane perpendicular to the curve (perpendicular to  $[x'(t), y'(t), z'(t)]$ ).

#3:  $\text{CURVE\_TO\_TUBE}(v, r) := v + \text{ROTATE\_Z}\left(\text{ATAN}\left(\frac{d}{dt} v_2, \frac{d}{dt} v_1\right)\right)$

$$\left. \begin{matrix} v \\ 1 \end{matrix} \right) \cdot \text{ROTATE\_Y}\left(\text{ACOS}\left(\frac{\left|\frac{d}{dt} v_3\right|}{\left|\frac{d}{dt} v\right|}\right)\right) \cdot [r \cdot \cos(s), r \cdot \sin(s), 0]$$

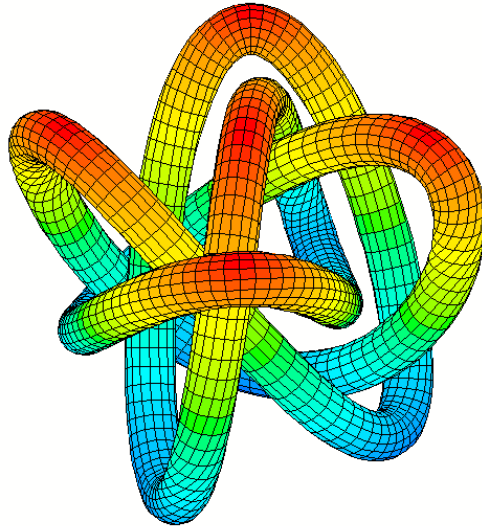
#4: In LISSAJOUS\_3D\_TUBE(a,b,c,r), a,b,c = x,y,z sine wave cycles at t goes from  $-\pi$  to  $\pi$ , and r = tube radius. All generated figures fit in a 5x5x5 box centered at the origin.

#5:  $\text{LISSAJOUS\_3D\_TUBE}(a, b, c, r) := \text{CURVE\_TO\_TUBE}([(5 - r) \cdot \sin(a \cdot t), (5 - r) \cdot \sin(b \cdot t), (5 - r) \cdot \cos(c \cdot t)], r)$

#6: Plot with factory defaults + Approximate Before Plotting + Plot Color to Rainbow + 3D Axes, Box, and Legend all off + s and t ranges  $-\pi$  to  $\pi$ .

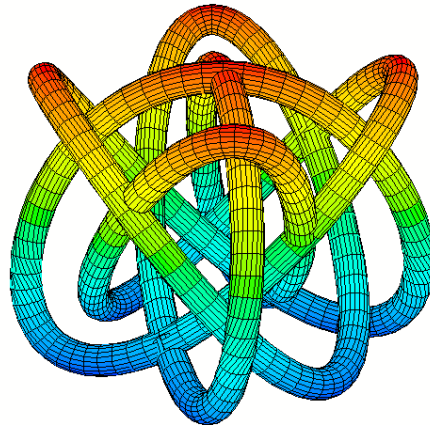
#7: 16 s panels, 300 t panels (my favorite):

#8:  $\text{LISSAJOUS\_3D\_TUBE}(3, 4, 5, 0.7)$



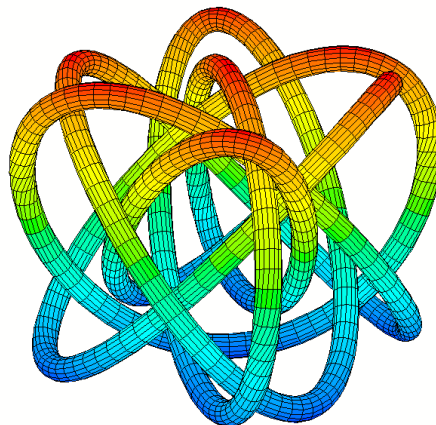
#9: This one intersects itself:

#10: `LISSAJOUS_3D_TUBE(4, 5, 6, 0.5)`



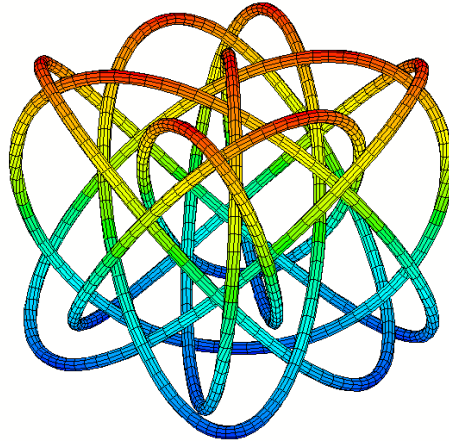
#11: 16 s panels, 400 t panels:

#12: `LISSAJOUS_3D_TUBE(5, 6, 7, 0.4)`



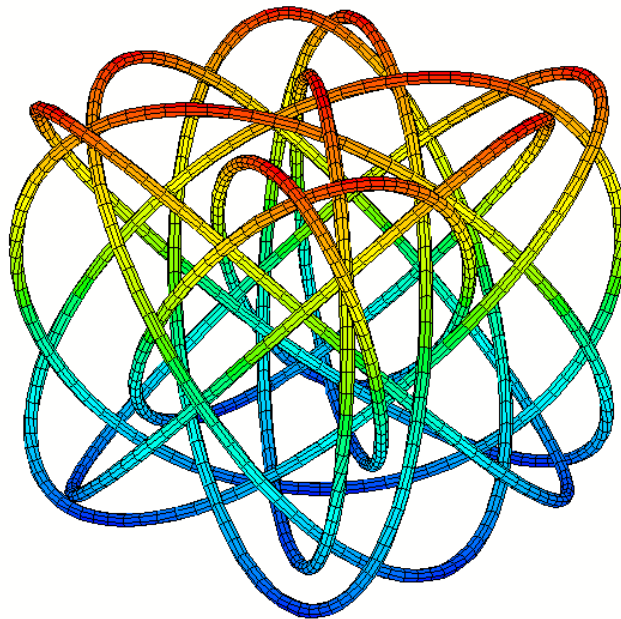
#13: 10 s panels, 600 t panels:

#14: LISSAJOUS\_3D\_TUBE(7, 8, 9, 0.2)



#15: 8 s panels, 800 t panels:

#16: LISSAJOUS\_3D\_TUBE(9, 10, 11, 0.15)

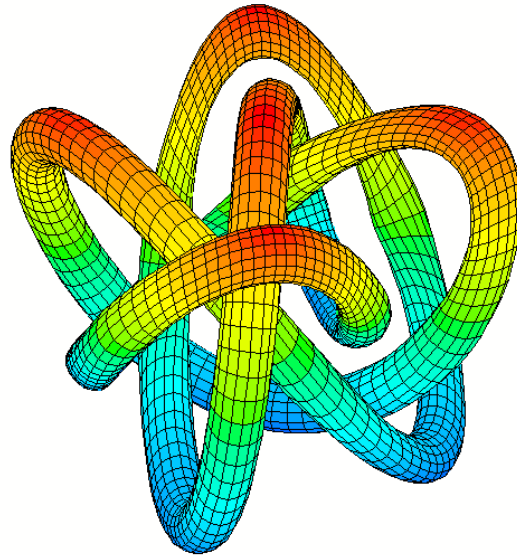


#17: Graphics.mth function SPACE\_TUBE makes a different result, with the tube rings not perpendicular to the curve being tubed. But the lines inline with the tube look better, not being 'twisted' around tight turns.

#18: LOAD(DfwMath\Graphics.mth)

#19: LISSAJOUS\_3D\_TUBE2(a, b, c, r) := SPACE\_TUBE([(5 - r)·SIN(a·t), (5 - r)·SIN(b·t), (5 - r)·COS(c·t)], t, r, φ)

#20: LISSAJOUS\_3D\_TUBE2(3, 4, 5, 0.7)



### Editor's Comments:

There are more problems with SPACE\_TUBE from GRAPHICS.MTH: Inspecting the plot above you might recognize a minor irregularity in the part right above. I tried d to produce a cylinder as a tube based on a straight line:

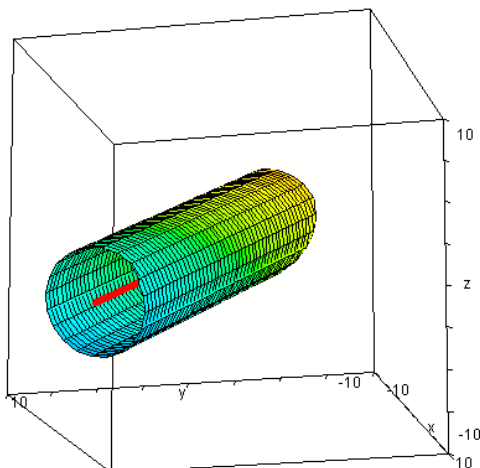
$$\text{SPACE\_TUBE}\left(\left[2 \cdot t + 1, 3 - \frac{t}{2}, t\right], t, 3, s\right)$$

$$\left[2 \cdot t + 1, 3 - \frac{t}{2}, t\right] + (3 \cdot \cos(s) + 3 \cdot \sin(s)) \cdot \text{SIGN}([0, 0, 0])$$

This does not work! Now let's apply Rick's function:

$$\text{CURVE\_TO\_TUBE}\left(\left[2 \cdot t + 1, 3 - \frac{t}{2}, t\right], 3\right)$$

$$\text{VECTOR}\left(\left[\left[2 \cdot t + 1, 3 - \frac{t}{2}, t\right]\right], t, -5, 5, 0.01\right)$$



The VECTOR-command generates the “core” of the tube as a thick line. It works properly.

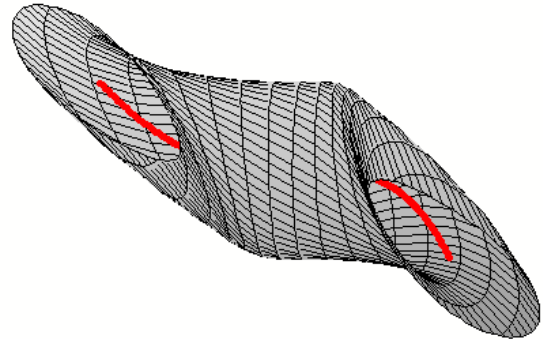
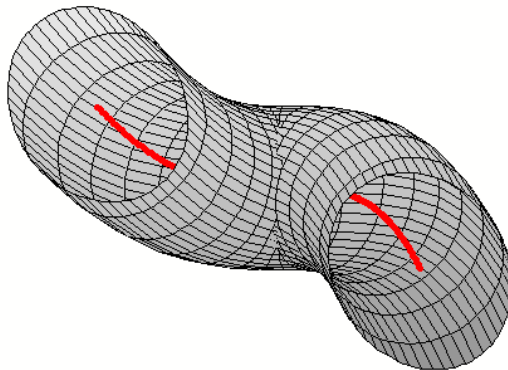
And here is another example to compare both methods:

$$\text{spc} := \left[ \begin{array}{c} 2 \\ 2 \cdot t^2 + 1, -2 \cdot t, \frac{t^3}{2} \end{array} \right]$$

SPACE\_TUBE(spc, t, 1.5, s)

VECTOR([spc], t, -5, 5, 0.01)

CURVE\_TO\_TUBE(spc, 1.5)



I can imagine that Nspire users would like to present tubes on a 3D-Graphing screen in the Graphs & Geometry application.

I start with Rick's version. DERIVE's ATAN(y,x)-function is not available on the Nspire, so it must be written in an extended form (argument in rotate\_z()). Parameters are  $t$  and  $u$  by default.

Functions ported from DERIVE:

$$\text{rotate\_z}(\alpha) := \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{rotate\_y}(\beta) := \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$\text{c\_t\_t}(v,r) := v + \text{rotate\_z} \left( \frac{\pi}{2} \cdot \text{sign} \left( \frac{d}{dt}(v[2 \ 1]) \right) - \tan^{-1} \left( \frac{\frac{d}{dt}(v[1 \ 1])}{\frac{d}{dt}(v[2 \ 1])} \right) \right) \cdot \text{rotate\_y} \left( \cos^{-1} \left( \frac{\frac{d}{dt}(v[3 \ 1])}{\text{norm} \left( \frac{d}{dt}(v) \right)} \right) \right) \cdot \begin{bmatrix} r \cdot \cos(u) \\ r \cdot \sin(u) \\ 0 \end{bmatrix}$$

$$\text{spc} := \left[ \begin{array}{c} 2 \cdot t^2 + 1 \\ -2 \cdot t \\ \frac{t^3}{2} \end{array} \right]$$

$$\text{c\_t\_t}(\text{spc}, 3) \rightarrow \begin{bmatrix} \frac{18 \cdot t^3 \cdot \cos(u)}{\sqrt{(4 \cdot t^2 + 1) \cdot (9 \cdot t^4 + 64 \cdot t^2 + 16)}} + \frac{3 \cdot \sin(u)}{\sqrt{4 \cdot t^2 + 1}} + 2 \cdot t^2 + 1 \\ \frac{-9 \cdot t^2 \cdot \cos(u)}{\sqrt{(4 \cdot t^2 + 1) \cdot (9 \cdot t^4 + 64 \cdot t^2 + 16)}} + \frac{6 \cdot t \cdot \sin(u)}{\sqrt{4 \cdot t^2 + 1}} - 2 \cdot t \\ \frac{t^3}{2} - \frac{3 \cdot \sqrt{16 \cdot (4 \cdot t^2 + 1)} \cdot \cos(u)}{\sqrt{9 \cdot t^4 + 64 \cdot t^2 + 16}} \end{bmatrix}$$

The components of the result must be transferred to the 3D Graph Entry/Edit > Parametric xp, yp, zp-entries. Then adjust the range and use the attributes as you like.

The second figure is the cylinder from above. It is no problem to introduce sliders for one or more parameters of the cylinder. Be aware to simplify the function first before setting the sliders in the 3D-Graphing screen.

Cylinder:

$$\text{cyl}(t,u) := \text{c\_t\_t}\left(2 \cdot t \quad t-2 \quad 3-\frac{t}{2}, 1\right) \quad \blacktriangleright \text{Done}$$

$$\text{cyl}(t,u) \blacktriangleright \begin{bmatrix} 2 \cdot t - \frac{2 \cdot \sqrt{105} \cdot \cos(u)}{105} - \frac{\sqrt{5} \cdot \sin(u)}{5} \\ t - \frac{\sqrt{105} \cdot \cos(u)}{105} + \frac{2 \cdot \sqrt{5} \cdot \sin(u)}{5} - 2 \\ \frac{-t}{2} - \frac{2 \cdot \sqrt{105} \cdot \cos(u)}{21} + 3 \end{bmatrix}$$

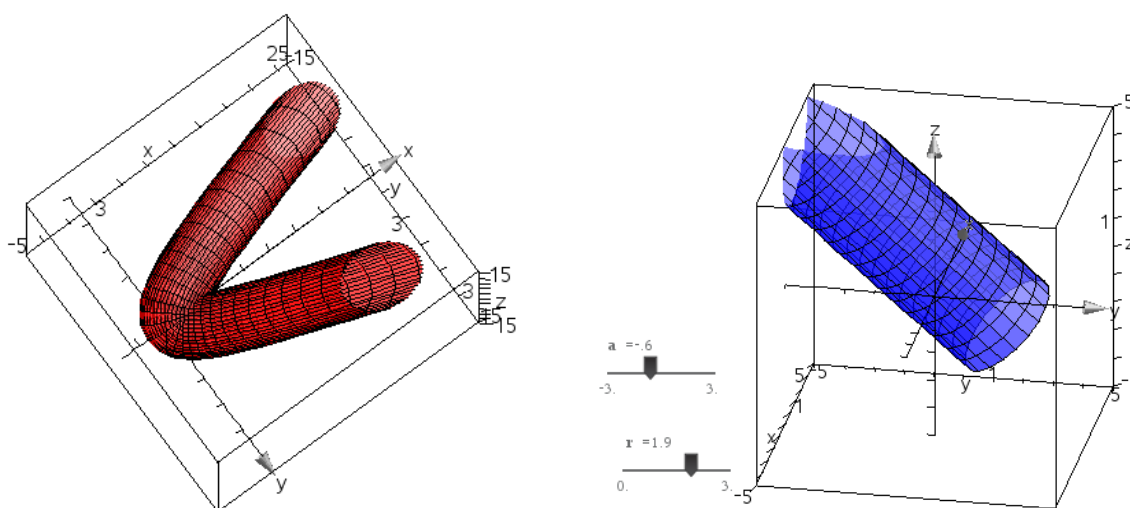
Cylinder with variables (Sliders):

$$\text{c\_t\_t}\left(a \cdot t \quad t-2 \quad 3-\frac{t}{2}, r\right)$$

The Lissajous-Tubes:

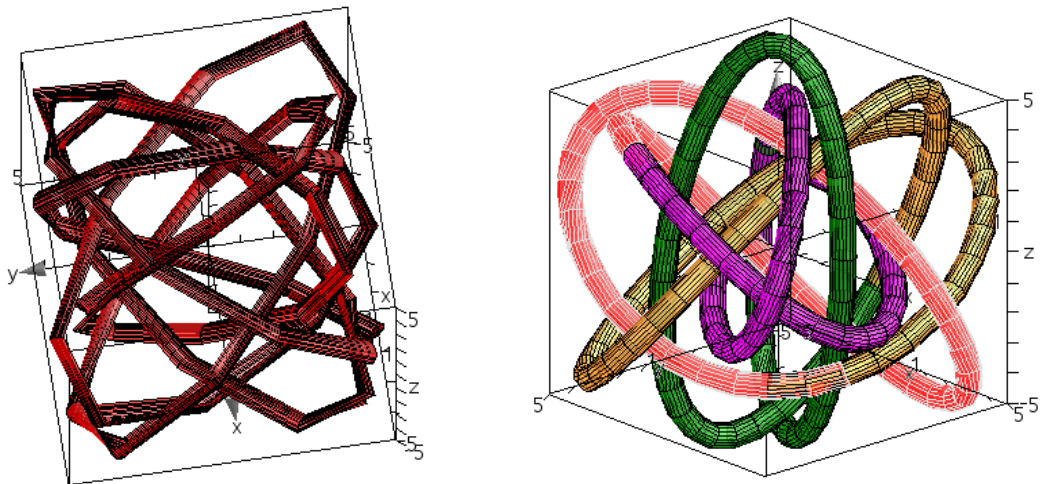
$$\text{liss\_tube}(a,b,c,r) := \text{c\_t\_t}\left((5-r) \cdot \left[\sin(a \cdot t) \quad \cos(b \cdot t) \quad \cos(c \cdot t)\right], r\right) \quad \blacktriangleright \text{Done}$$

$$\text{liss\_tube}(5,6,7,4/10)$$

$$\left(14 \cdot \sin(7 \cdot t) \cdot \cos\left(\frac{\pi \cdot \text{sign}(\sin(6 \cdot t))}{2} - \tan^{-1}\left(\frac{5 \cdot \cos(5 \cdot t)}{6 \cdot \sin(6 \cdot t)}\right)\right) \cdot \cos(u) - \sqrt{49 \cdot (\sin(7 \cdot t))^2 + 36 \cdot \dots}\right)$$


Plot of the tube around space curve spc from above (left) and of the cylinder with sliders for  $a$  and radius  $r$ .

Now let's try to create Rick's the Lissajous tube on the Nspire screen.



As maximum resolution for  $t$  is 50 we receive a not really nice plot (left). We can improve the figure by splitting it into four parts with  $t$  from  $-\pi$  to  $-\pi/2$ , from  $-\pi/2$  to 0 etc. This will give the plot on the right-hand side. I used different colors for the single parts in order to show the single parts.

Finally, I wanted to define the original DfW5 space\_tube function for the TI-Nspire. DERIVE's SIGN-function does not only return the sign of a number, but the normalized vector for a vector.  $\mathbf{sg}(\mathbf{v}_-)$  is my respective Nspire-function.

You can see that it not possible to create a cylinder as the tube of a straight line

Original space\_tube-function from DERIVE:

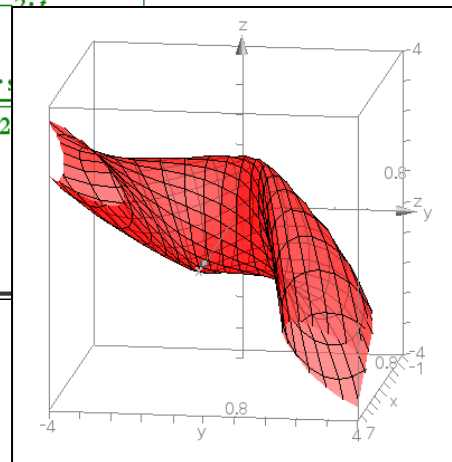
$$\mathbf{sg}(\mathbf{v}_-) := \text{when}(\text{getType}(\mathbf{v}_-) = \text{"MAT"}, \frac{\mathbf{v}_-}{\text{norm}(\mathbf{v}_-)}, \text{sign}(\mathbf{v}_-)) \quad \blacktriangleright \text{Done}$$

$$\text{normal\_vector}(\mathbf{v}_-, t) := \mathbf{sg}\left(\frac{d^2}{dt^2}(\mathbf{v}_-)\right) \quad \blacktriangleright \text{Done} \quad \text{binormal}(\mathbf{v}_-, t) := \mathbf{sg}\left(\text{crossP}\left(\frac{d}{dt}(\mathbf{v}_-), \frac{d^2}{dt^2}(\mathbf{v}_-)\right)\right) \quad \blacktriangleright \text{Done}$$

$$\text{space\_tube}(\mathbf{v}, r) := \mathbf{v} + r \cdot \sin(u) \cdot \text{normal\_vector}(\mathbf{v}, t) + r \cdot \cos(u) \cdot \text{binormal}(\mathbf{v}, t) \quad \blacktriangleright \text{Done}$$

$$\text{space\_tube}\left(\begin{bmatrix} 2 \cdot t^2 + 1 & -2 \cdot t & \frac{t^3}{2} \end{bmatrix}, 1.5\right) \blacktriangleright \begin{bmatrix} \frac{-4.5 \cdot t \cdot \cos(u)}{\sqrt{9 \cdot t^4 + 9 \cdot t^2 + 16}} + \frac{6 \cdot \sin(u)}{\sqrt{9 \cdot t^2 + 16}} + 2 \cdot t^2 + 1 \\ \frac{-4.5 \cdot t^2 \cdot \cos(u)}{\sqrt{9 \cdot t^4 + 9 \cdot t^2 + 16}} - 2 \cdot t \\ \frac{6 \cdot \cos(u)}{\sqrt{9 \cdot t^4 + 9 \cdot t^2 + 16}} + \frac{4.5 \cdot t \cdot \sin(u)}{\sqrt{9 \cdot t^2 + 16}} \end{bmatrix}$$

$$\text{space\_tube}\left(\begin{bmatrix} 3 \cdot t - 1 & \frac{t}{2} & 1 - t \end{bmatrix}, 2\right) \blacktriangleright \begin{bmatrix} \text{undef} \\ \text{undef} \\ \text{undef} \end{bmatrix}$$



This is the same strange space tube as above.

## Mein Mathematik-Abitur

Wolfgang Alvermann, GER

Bei einem Klassentreffen 2013, 45 Jahre nach unserem Abitur 1968, brachte uns unser geschätzter Mathematiklehrer die damalige Abitur-Klausur mit, die wir, die Klasse 13m des mathematisch-naturwissenschaftlichen Zweigs des *NEUEN GYMNASIUMS OLDENBURG* zu schreiben hatten.

Ein DIN-A4-Blatt reichte aus, um uns 5 Stunden zu beschäftigen und inhaltlich zu fordern.

Hilfsmittel: Formelsammlung, Geo-Dreieck, Zirkel, Bleistift

### Aufgaben:

- (1) Ein Sektkelch (ohne Fußteller) entsteht durch Rotation einer Hyperbel mit den Asymptoten um die x-Achse. Die Kelchöffnung hat einen äußeren Durchmesser von 6cm. Die Gesamthöhe des Kelches beträgt 18cm, der Hohlraum ist 12cm tief.
- Konstruieren Sie zwei Hyperbelpunkte mit  $r_1 = 1$  bzw.  $r_1 = 6$ .
  - Wie viel wiegt der leere Kelch? ( $\rho_{\text{Glas}} = 2,5 \frac{\text{g}}{\text{cm}^3}$ ).
  - Wie viel Liter fasst das Glas, wenn es
    - a) randvoll,
    - b) bis 3cm unterhalb des Kelchrandes (auf der Kelchachse gemessen) gefüllt ist?
  - Wie viel mm beträgt die Wandstärke des Glases an der Kelchöffnung?
- (2) Für welchen Punkt P der Kurve  $y = \ln(x)$  nimmt das Subnormalendreieck einen extremalen Flächeninhalt an? (Unter dem Subnormalendreieck versteht man dasjenige Dreieck, das von der x-Achse, der Kurvennormalen und dem Lot vom Kurvenpunkt auf die x-Achse begrenzt wird.)
- Stellen Sie zunächst eine Vorbetrachtung an.
  - Gegen welchen Grenzwert strebt der Flächeninhalt des Dreiecks, wenn die Abszisse des Kurvenpunktes  $x_1 \rightarrow \infty$  strebt? (Wählen Sie dazu eine geeignete Substitution  $x_1 = g(z_1)$ )
- (3) Welche Parabel mit dem Scheitelpunkt B(a/0) schneidet die gegebene Parabel  $y^2 = 2px + 2pa$  senkrecht?
- Bestimmen Sie die Koordinaten des Schnittpunktes und die Lage der beiden Parabelbrennpunkte.
  - Auf welcher Kurve bewegt sich der Schnittpunkt der Parabeln, wenn sich der Parameter p ändert, die Scheitelpunkte der Parabel jedoch festbleiben.
  - Zeichnen Sie ein Parabelpaar für a=2 und p=2 sowie die gefundene Ortslinie.



## My Mathematics Final Exam (Abitur)

At a class reunion in 2013, 45 years after our final exam (Abitur) in 1968, our high appreciated mathematics teacher brought our mathematics written exam to us, which we, class 13m of the mathematical-scientific branch of the NEUEN GYMNASIUM OLDENBURG had to write.

One DIN-A4 sheet was sufficient to keep us busy for demanding five hours.

Permitted tools: formulary, Geo-triangle, compass, pencil

### The Tasks:

- (1) A champagne cup (without foot plate) is formed by a rotation of a hyperbola together with its asymptotes about the  $x$ -axis. The outer diameter of the cup opening is 6 cm, the total height of the cup is 18 cm and its hollow is 12 cm deep.
  - Construct two points of the hyperbola with  $r_1 = 1$ , resp.  $r_1 = 6$ .
  - What is the weight of the cup ( $\rho_{\text{glass}} = 2.5\text{g/cm}^3$ )?
  - How many liters does the glass hold, when it is
    - a) half full,
    - b) filled until 3 cm below the cup rim (measured on the axis)?
  - How thick is the glass at the cup opening?
  
- (2) For which point on the graph of  $y = \ln(x)$  gives the subnormal-triangle the maximum area? (The subnormal triangle is formed by the  $x$ -axis, the normal of the curve and the vertical line through the point.)
  - Make a preliminary observation.
  - What is the limit of the area of this triangle when the  $x$ -value of the point  $x_1$  tends to  $\infty$ ? (Choose an appropriate substitution  $x_1 = g(z_1)$ .)
  
- (3) Which parabola with its vertex  $B(a/0)$  intersects the given parabola  $y^2 = 2px + 2pa$  vertically?
  - Find the coordinates of the intersection point of the two parabolas and the position of their foci.
  - What is the locus of the intersection points of the parabolas if parameter  $p$  changes, but their vertices remain fixed?
  - Draw one pair of parabolas for  $a = 2$  and  $p = 2$  together with the locus.

### Editor's Comment:

On the next pages you will find Wolfgang's treatment of the tasks with paper and pencil how it was done "several" years ago. I leave the German version because I believe that you can follow his procedure.

I thought that it might be interesting to tackle the problems with TI-Nspire (or DERIVE or any other technology). This is done with English comments beginning on page 36.

**Lösung Aufgabe 1:***Bestimmung der Hyperbelgleichung*

Die Hyperbel hat die Gleichung  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  mit den Asymptoten  $y = \pm \frac{b}{a} \cdot x$

Mit  $a = 6$  und dem geg. Punkt  $P = (18;3)$  errechnet man  $b: 3 = \frac{b}{6} \cdot 18$   **$b = 1$**

Die Hyperbel hat also die Form  $\frac{x^2}{36} - y^2 = 1$  bzw.  $y^2 = \frac{x^2}{36} - 1$

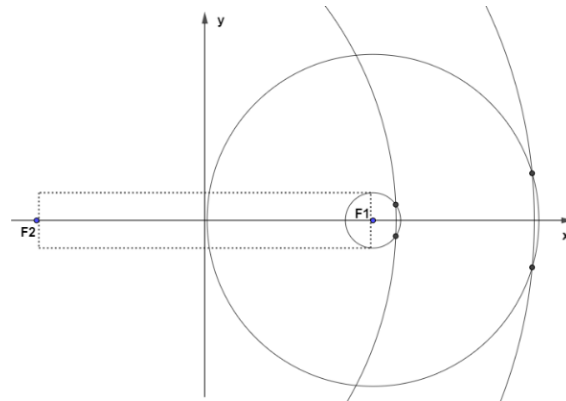
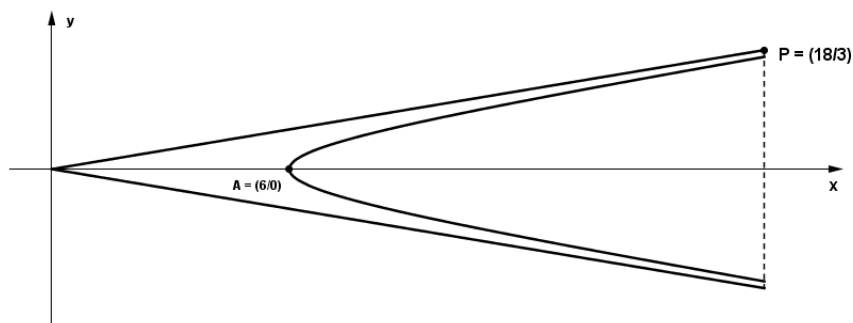
*Zeichnung 1: Konstruktion der Punkte*

Mit  $e = \sqrt{a^2 + b^2} = \sqrt{37}$  erhält man die Brennpunkte:

$$F_1 = (\sqrt{37}; 0)$$

$$F_2 = (-\sqrt{37}; 0)$$

und konstruiert mit Kreisen um  $F_1$  mit  $r_1 = 1$  und  $F_2$  mit  $r_2 = 13$  bzw. um  $F_1$  mit  $r_1 = 6$  und  $F_2$  mit  $r_2 = 18$  die gesuchten Punkte.

*Zeichnung 2: Die Hyperbel mit den Asymptoten**Masse des leeren Kelches.*

Berechnet werden zunächst die Rotationsvolumina der Asymptote  $V_1$  und der Hyperbel  $V_2$ .  
[Ergebnisse in  $\text{cm}^3$  bzw. Gramm]:

$$V_1 = \frac{1}{3} \cdot \pi \cdot 3^2 \cdot 18 = 54 \cdot \pi$$

$$V_2 = \pi \cdot \int_6^{18} \left( \frac{x^2}{36} - 1 \right) dx = 40 \cdot \pi$$

$$\text{Masse: } m = V_{\text{ges}} \cdot \rho_{\text{Glas}} = (V_1 - V_2) \cdot \rho_{\text{Glas}} = 14 \cdot \pi \cdot 2.5$$

$$\underline{m = 35 \cdot \pi \approx 110 \text{ g}}$$

*Füllmengen.*

$$\text{Vollständig gefüllt} = V_2: V_2 = \pi \cdot \int_6^{18} \left( \frac{x^2}{36} - 1 \right) dx = 40 \cdot \pi$$

$$\underline{V_2 \approx 125 \text{ cm}^3}$$

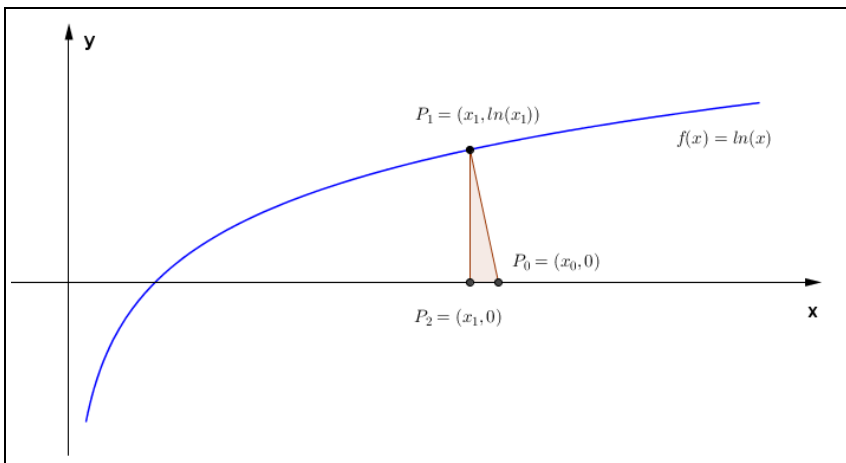
3 cm unterhalb des Randes:  $V_3 = \pi \cdot \int_6^{15} \left( \frac{x^2}{36} - 1 \right) dx = \frac{81}{4} \cdot \pi$   $V_3 \approx 63.6 \text{ cm}^3$

Wandstärke:

$x = 18$  in  $y^2 = \frac{x^2}{36} - 1$  einsetzen ergibt:  $y = 2 \cdot \sqrt{2} \rightarrow$   $s \approx 0.172 \text{ cm}$

### Lösung Aufgabe 2:

Zeichnung:



Bestimmung der Normalen im Punkt  $P_1$  und von  $x_0$ :

Steigung der Tangente im Punkt  $P_1$ :  $f'(x) = \frac{1}{x} \rightarrow m_t = f'(x_1) = \frac{1}{x_1}$

Steigung der Normalen im Punkt  $P_1$ :  $m_n = -x_1$

Gerade durch  $P_1$  und  $P_0$   $\left. \begin{array}{l} y = m \cdot x + b \\ \ln(x_1) = -x_1 \cdot x_1 + b \end{array} \right\} \rightarrow b = x_1^2 + \ln(x_1)$

$$y = -x_1 \cdot x + x_1^2 + \ln(x_1)$$

Mit  $y = 0$  folgt:  $x_0 = x_1 + \frac{\ln(x_1)}{x_1}$

Für die Dreiecksfläche folgt:  $A(x_1) = \frac{1}{2} \cdot \frac{\ln^2(x_1)}{x_1}$

$$A'(x_1) = \frac{1}{2} \cdot \frac{2 \cdot \ln(x_1) - \ln^2(x_1)}{x_1^2}$$

Diese Fläche soll extremal werden:

$$A''(x_1) = \frac{\ln^2(x_1) - 3 \cdot \ln(x_1) + 1}{x_1^3}$$

Daraus folgt:  $P_0 = (e^2 / 2)$

$A = \frac{2}{e^2}$  Maximum weil  $A''(e^2) < 0$

Grenzwert des Flächeninhalts

$$\lim_{x_1 \rightarrow \infty} A(x_1) = 0!$$

Begründung: Betrachtet wird mit der Substitution  $x_1 = e^{z_1}$  der Term  $A(z_1) = \frac{\ln^2(e^{z_1})}{e^{z_1}} = \frac{z_1^2}{e^{z_1}}$

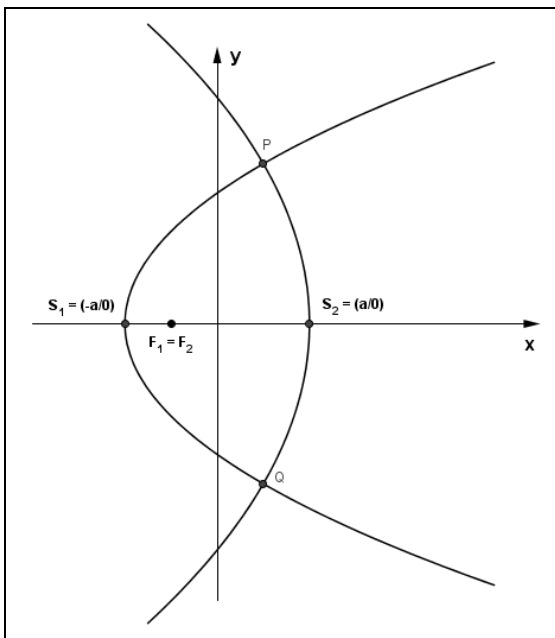
Da die Exponentialfunktion schneller wächst als jede andere Funktion, ergibt sich die Behauptung.

Anderer Weg: Regel von de L'Hospital

$$\lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} = \lim_{x \rightarrow \infty} \frac{g'(x)}{h'(x)} = \lim_{x \rightarrow \infty} \frac{g''(x)}{h''(x)}$$

$$A(x_1) = \frac{1}{2} \cdot \frac{\ln^2(x_1)}{x_1}; \quad \frac{g'(x_1)}{h'(x_1)} = \frac{\ln(x_1) \cdot \frac{1}{x_1}}{1} = \frac{\ln(x_1)}{x_1} \quad \frac{g''(x_1)}{h''(x_1)} = \frac{\frac{1}{x_1}}{1} = \frac{1}{x_1}$$

**Lösung Aufgabe 3:**



Die Parabelgleichungen:

$$p_1: y^2 = 2 \cdot p \cdot (x + a) \quad y = \pm \sqrt{2p \cdot (x + a)}$$

$$p_2: y^2 = -2 \cdot q \cdot (x - a) \quad y = \pm \sqrt{-2q \cdot (x - a)}$$

Im Schnittpunkt  $P = (x_1/y_1)$  der beiden Parabeln sind die Steigungen orthogonal.

Betrachtet werden die oberen Teilstücke der Parabeln mit ihren Ableitungen:

$$p_1: y' = \frac{p}{\sqrt{2p \cdot (x + a)}} = \frac{p}{y}$$

$$p_2: y' = \frac{-q}{\sqrt{2q \cdot (x - a)}} = \frac{-q}{y}$$

Steigungen:

$$\left. \begin{array}{l} m_1 = \frac{p}{y_1} \\ m_2 = \frac{-q}{y_2} \end{array} \right\} \text{ mit } y_1 = y_2 \text{ folgt aus } m_1 \cdot m_2 = -1: p \cdot q = y_1^2 \text{ bzw. } p \cdot q = y_2^2$$

$$\left. \begin{array}{l} p \cdot q = 2p \cdot (x_1 + a) \\ p \cdot q = -2q \cdot (x - a) \end{array} \right\} \rightarrow p + q = 4a \quad q = 4a - p$$

Schnittpunkt der beiden Parabeln

$$2 \cdot p \cdot (x + a) = -2 \cdot q \cdot (x - a) \quad \underline{P = \left( a - \frac{p}{2} / \sqrt{p \cdot (4a - p)} \right)}$$

Parabelbrennpunkte:

Für Parabeln gilt:

Abstand des Punktes S zur Leitlinie = Abstand von S zu F = p/2

$$F_1 = \left( \frac{p}{2} - a / 0 \right) \quad F_2 = \left( -\frac{q}{2} + a \right) = \left( -\frac{4a - p}{2} + a / 0 \right) = \left( \frac{p}{2} - a / 0 \right) \quad \boxed{F_1 = F_2}$$

Ortslinie der Schnittpunkte mit festem a

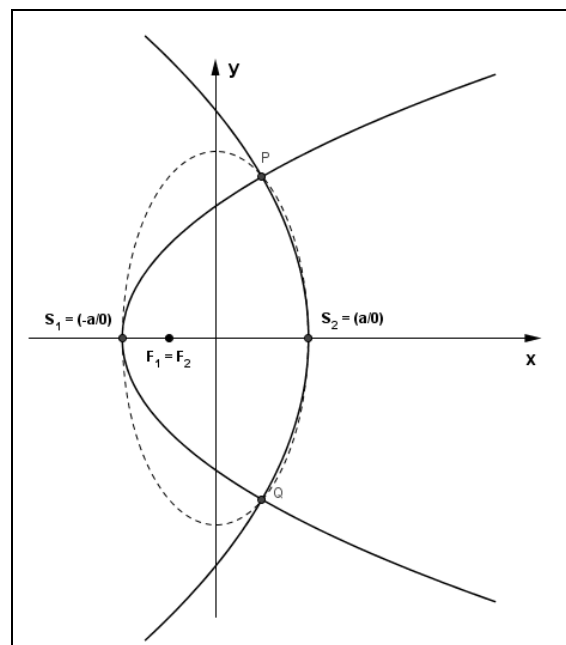
Aus dem Schnittpunkt muss p eliminiert werden; mit  $x = a - \frac{p}{2}$  (x-Wert des SP) folgt

$p = 2 \cdot (a - x)$  Eingesetzt in den y-Wert des Schnittpunktes errechnet man die Ortslinie.

$$\left. \begin{array}{l} y = \sqrt{2 \cdot (a - x) \cdot (4a - 2a + 2x)} \\ y = \sqrt{2 \cdot (a - x) \cdot 2 \cdot (a + x)} \end{array} \right\} \rightarrow \boxed{y = 2 \cdot \sqrt{a^2 - x^2}}$$

Umgeformt erhält man eine Ellipse der Form

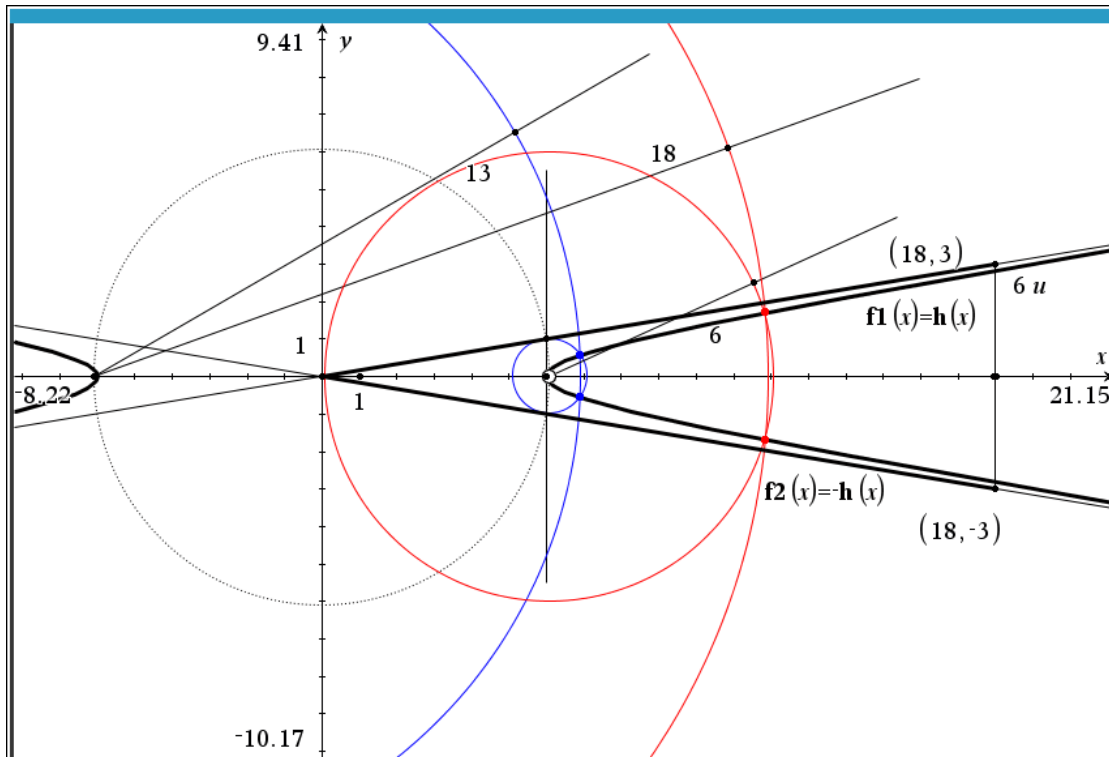
$$\frac{x^2}{a^2} + \frac{y^2}{4a^2} = 1 \text{ mit den Halbachsen } a \text{ und } 2a.$$



A possible TI-Nspire treatment of the problems:

Task 1:

Construction of the points (blue and red) and sketch of the complete figure in one graph:



Problem 1: Axes of the hyperbola (centre = (0,0))

$$a=6 \rightarrow 6 \quad \text{solve}\left(\frac{18}{3}=\frac{6}{b}, b\right) \rightarrow b=1 \quad b=1 \rightarrow 1$$

Hyperbola:  $x^2/36 - y^2 = 1$

$$\text{solve}\left(\frac{x^2}{36} - y^2 = 1, y\right) \rightarrow y = \frac{\sqrt{x^2 - 36}}{6} \text{ and } x^2 - 36 \geq 0 \text{ or } y = \frac{-\sqrt{x^2 - 36}}{6} \text{ and } x^2 - 36 \geq 0$$

$$g(x) = \frac{3}{18} \cdot x \rightarrow \text{Done} \quad h(x) = \frac{\sqrt{x^2 - 36}}{6} \rightarrow \text{Done}$$

$$\text{Weight of the empty champagne flute: } \pi \cdot \left( \int_0^{18} (g(x))^2 dx - \int_6^{18} (h(x))^2 dx \right) \cdot 2.5 \rightarrow 35 \cdot \pi g$$

$$\pi \cdot \int_6^{18} (h(x))^2 dx \rightarrow 40 \cdot \pi = 125.66 \text{ cm}^3 \quad \pi \cdot \int_6^{15} (h(x))^2 dx \rightarrow \frac{81 \cdot \pi}{4} = 63.62 \text{ cm}^3$$

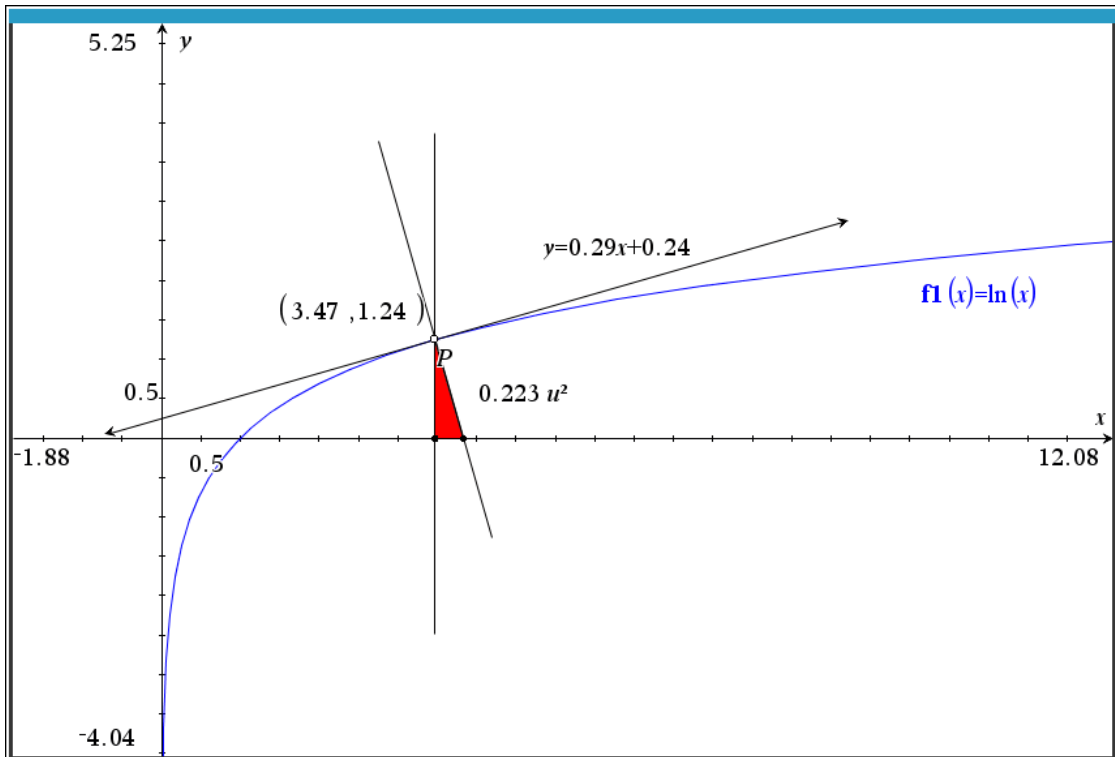
Thickness of the glass:

$$g(18) - h(18) \rightarrow 3 - 2 \cdot \sqrt{2} = 0.17 \text{ cm}$$

□

Task 2:

Construction of the points (blue and red) and sketch of the complete figure in one graph:



You can follow the procedure in the Notes:

Normal:  $\text{norml} := y - \ln(t) = -t \cdot (x-t) \rightarrow y - \ln(t) = t \cdot (t-x)$

solve( $y=0$  and  $\text{norml}, \{x,y\}$ )  $\rightarrow x = \frac{\ln(t)+t^2}{t}$  and  $t > 0$  and  $y=0$

$a(t) := \frac{1}{2} \cdot \ln(t) \cdot \left( \frac{\ln(t)+t^2}{t} - t \right) \rightarrow \text{Done}$        $a(t) \rightarrow \frac{(\ln(t))^2}{2 \cdot t}$

solve( $\frac{d}{dt}(a(t))=0, t$ )  $\rightarrow t=1$  or  $t=e^2$  ⚠       $\frac{d^2}{dt^2}(a(t))|_{t=\{1, e^2\}} \rightarrow \{1, -e^{-6}\}$  ⚠

Maximum for  $x = e^2$ . Point  $\{e^2, \ln(e^2)\} \rightarrow \{e^2, 2\}$

Maximum Area:  $a(e^2) \rightarrow 2 \cdot e^{-2} = 0.27$

Limit:  $\lim_{t \rightarrow \infty} (a(t)) \rightarrow 0$

Reasoning:  $\ln(t)$  grows slower than all other functions  $\rightarrow a(t)$  tends to 0-

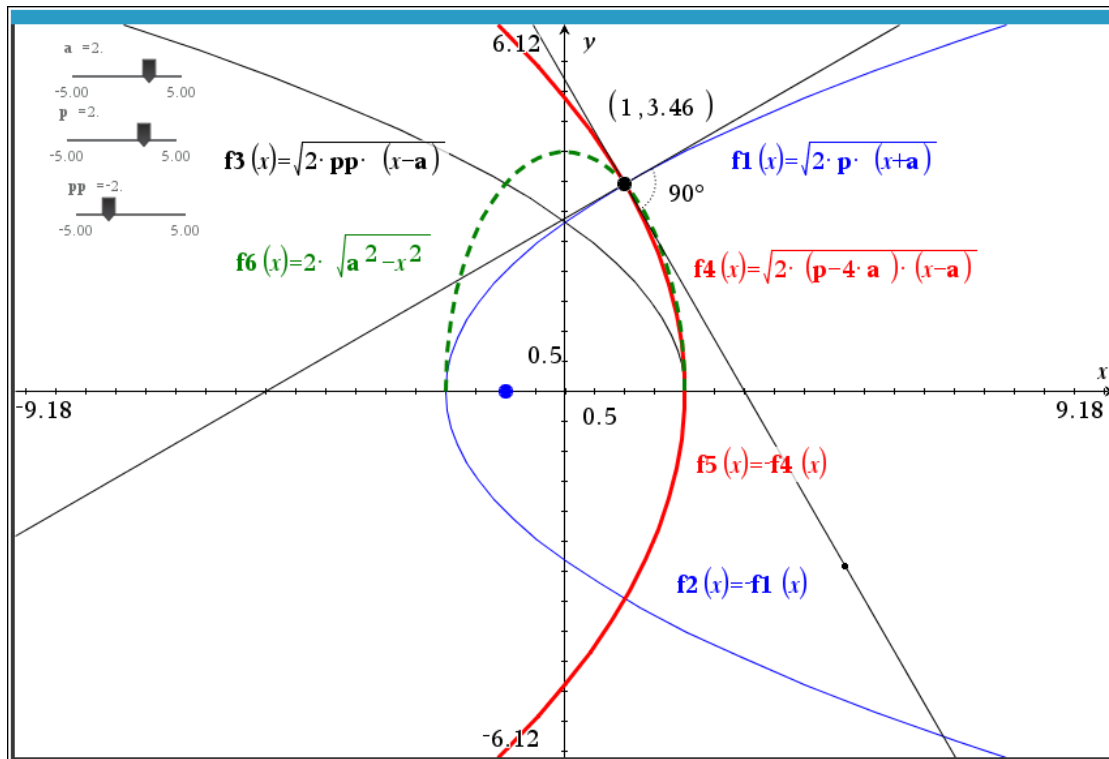
or: Substitution

$a(t)|_{t=e^z} \rightarrow \frac{z^2 \cdot e^{-z}}{2} = \frac{z^2}{2 \cdot e^z}$   $e^z$  grows faster than all other functions  $\rightarrow g(z)$  tends to 0.

## Task 2:

This is the final graphic representation (with some extras added).

I added the tangents in one intersection point (perpendicular) and could not resist to install sliders for parameters a, p and pp.



$$p1: p1: y^2 = 2 \cdot p_- \cdot (x+a_-) \rightarrow y^2 = 2 \cdot p_- \cdot (x+a_-)$$

$$p2: p2: y^2 = 2 \cdot pp_- \cdot (x-a_-) \rightarrow y^2 = 2 \cdot pp_- \cdot (x-a_-)$$

solve(p1 and p2, {x,y})

$$\rightarrow x = \frac{-a_- \cdot (p_- + pp_-)}{p_- - pp_-} \text{ and } y = \frac{2 \cdot \sqrt{a_-} \cdot \sqrt{p_-} \cdot \sqrt{pp_-}}{\sqrt{p_- - pp_-}} \text{ or } x = \frac{-a_- \cdot (p_- + pp_-)}{p_- - pp_-} \text{ and } y = \frac{-2 \cdot \sqrt{a_-} \cdot \sqrt{p_-} \cdot \sqrt{pp_-}}{\sqrt{p_- - pp_-}}$$

$$s11 := \frac{d}{dx} \left( \sqrt{2 \cdot p_- \cdot (x+a_-)} \right) \Big|_{x = \frac{-a_- \cdot (p_- + pp_-)}{p_- - pp_-}} = \frac{p_-}{2 \cdot \sqrt{\frac{a_- \cdot p_- \cdot pp_-}{p_- - pp_-}}} \quad \Delta$$

$$s12 := \frac{d}{dx} \left( \sqrt{2 \cdot pp_- \cdot (x-a_-)} \right) \Big|_{x = \frac{-a_- \cdot (p_- + pp_-)}{p_- - pp_-}} = \frac{pp_-}{2 \cdot \sqrt{\frac{a_- \cdot p_- \cdot pp_-}{p_- - pp_-}}} \quad \Delta$$

$$s11 \cdot s12 = -1 \rightarrow \frac{-(p_- - pp_-)}{4 \cdot a_-} = -1 \quad \Delta$$

$$\text{solve} \left( \frac{-(p_- - pp_-)}{4 \cdot a_-} = -1, pp_- \right) \rightarrow pp_- = p_- - 4 \cdot a_-$$

$$\text{factor}(y^2 = 2 \cdot pp_- \cdot (x-a_-)) \Big|_{pp_- = p_- - 4 \cdot a_-} \rightarrow y^2 = -2 \cdot (4 \cdot a_- - p_-) \cdot (x-a_-)$$



Because of the sliders for a, p and pp it is necessary to work with a\_, p\_ and pp\_ in the calculation. Otherwise the values of the slider variables are taken for the calculations which results in confusion ...

factor( $y^2=2 \cdot pp\_ \cdot (x-a\_)$ ) |  $pp_=p_- - 4 \cdot a\_ \rightarrow y^2=-2 \cdot (4 \cdot a_- - p_-) \cdot (x-a_-)$

The parameter of the second parabola is  $4a-p$ .

The foci are:  $F1 = (-a + p/2, 0)$  (given parabola) and  $F2 = (a - (4a-p)/2, 0) = (-a+p/2, 0) = F1$

solve(p1 and p2, {x,y}) |  $pp_=p_- - 4 \cdot a_-$

$x = \frac{2 \cdot a_- - p_-}{2}$  and  $y = -\sqrt{4 \cdot a_- - p_-} \cdot \sqrt{p_-}$  or  $x = \frac{2 \cdot a_- - p_-}{2}$  and  $y = \sqrt{4 \cdot a_- - p_-} \cdot \sqrt{p_-}$  or

$x = \frac{c1^2}{2 \cdot p_-}$  and  $y = c1$  and  $a_- = 0$  or  $x = c2$  and  $y = 0$  and  $a_- = 0$  and  $p_- = 0$

Intersection points:  $((2a-p)/2, \pm \sqrt{(4a-p)p})$

solve( $x = \frac{2 \cdot a_- - p_-}{2}, p_-$ ) |  $p_- = 2 \cdot a_- - 2 \cdot x$

$y = -\sqrt{4 \cdot a_- - p_-} \cdot \sqrt{p_-}$  |  $p_- = 2 \cdot a_- - 2 \cdot x \rightarrow y = -2 \cdot \sqrt{x+a_-} \cdot \sqrt{a_- - x}$

$(y = -2 \cdot \sqrt{x+a_-} \cdot \sqrt{a_- - x})^2 \rightarrow y^2 = -4 \cdot (x+a_-) \cdot (x-a_-)$  ⚠

expand( $y^2 = -4 \cdot (x+a_-) \cdot (x-a_-)$ ) |  $y^2 = 4 \cdot a_-^2 - 4 \cdot x^2$

$y^2 + 4 \cdot x^2 = 4 \cdot a_-^2 \rightarrow 4 \cdot x^2 + y^2 = 4 \cdot a_-^2$

Ellipse: M(0,0); horiz. axis =  $2a$ , vertical axis =  $4a$

$p_- \quad ( -a \cdot (p_- + pp_-) )$

expand( $y^2 = -4 \cdot (x+a_-) \cdot (x-a_-)$ ) |  $y^2 = 4 \cdot a_-^2 - 4 \cdot x^2$

$y^2 + 4 \cdot x^2 = 4 \cdot a_-^2 \rightarrow 4 \cdot x^2 + y^2 = 4 \cdot a_-^2$

Ellipse: M(0,0); horiz. axis =  $2a$ , vertical axis =  $4a$

$y = -\sqrt{4 \cdot a_- - p_-} \cdot \sqrt{p_-} = \frac{p_-}{2 \cdot \sqrt{\frac{-a_- \cdot p_- \cdot pp_-}{p_- - pp_-}}} \cdot \left( x - \frac{-a_- \cdot (p_- + pp_-)}{p_- - pp_-} \right)$  |  $pp_=p_- - 4 \cdot a_-$

$\rightarrow y = -\sqrt{4 \cdot a_- - p_-} \cdot \sqrt{p_-} = \frac{p_- \cdot (2 \cdot x - 2 \cdot a_- + p_-)}{2 \cdot \sqrt{(4 \cdot a_- - p_-) \cdot p_-}}$  ⚠

The tangents in one intersection point:

$f7(x) = \sqrt{4 \cdot a_- - p_-} \cdot \sqrt{p_-} + \frac{p_-}{2 \cdot \sqrt{\frac{-a_- \cdot p_- \cdot pp_-}{p_- - pp_-}}} \cdot \left( x - \frac{-a_- \cdot (p_- + pp_-)}{p_- - pp_-} \right)$  |  $pp_=p_- - 4 \cdot a_-$  and  $p_-=p$  and  $a_-=a$

• Done

$f8(x) = \sqrt{4 \cdot a_- - p_-} \cdot \sqrt{p_-} + \frac{pp_-}{2 \cdot \sqrt{\frac{-a_- \cdot p_- \cdot pp_-}{p_- - pp_-}}} \cdot \left( x - \frac{-a_- \cdot (p_- + pp_-)}{p_- - pp_-} \right)$  |  $pp_=p_- - 4 \cdot a_-$  and  $p_-=p$  and  $a_-=a$

• Done

Mail from Rick Nungester (received April 16, 2020)

Josef,

I think we have a Derive defect that existed for many years but was fixed in Derive v6.01. Please answer the two questions below.

$\text{INT}(\text{ABS}(\text{SIN}(t) + \text{SIN}(2 \cdot t - \pi/2)), t, 0, 2 \cdot \pi)$

In Derive for DOS version 3.04 this simplifies to 0 (wrong).

In Derive for Windows v5.06 this simplifies to 0 (wrong).

In Derive for Windows v6.1 this simplifies to  $3 \cdot \sqrt{3}$ . **Is this true?**

The same waveform shifted in time gives me a different error.

$\text{INT}(\text{ABS}(\text{SIN}(t + \pi/6) - \text{COS}(2 \cdot t + \pi/3)), t, 0, 2 \cdot \pi)$

In Derive for DOS version 3.04 this simplifies to  $3 \cdot \sqrt{3}/2$  (wrong).

In Derive for Windows v5.06 this simplifies to  $3 \cdot \sqrt{3}/2$  (wrong).

In Derive for Windows v6.1 this simplifies to  $3 \cdot \sqrt{3}$ . **Is this true?**

Thank you,

Rick

For reference, **Derive for DOS v3.04 results.**

```

DOSBox 0.74, Cpu speed: 3000 cycles, Frameskip 0, Program: DERIVE
#1:  \int_0^{2 \cdot \pi} \left| \text{SIN}(t) + \text{SIN}\left[2 \cdot t - \frac{\pi}{2}\right] \right| dt
#2:  0
#3:  \int_0^{2 \cdot \pi} \left| \text{SIN}\left[t + \frac{\pi}{6}\right] - \text{COS}\left[2 \cdot t + \frac{\pi}{3}\right] \right| dt
#4:  \frac{3 \cdot \sqrt{3}}{2}
COMMAND: Author Build Calculus Declare Expand Factor Help Jump solve Manage
Options Plot Quit Remove Simplify Transfer Unremove move Window approx
Compute time: 2.3 seconds
Simp(#3)                               Free:100%                               Derive Algebra
  
```

Derive for Windows v5.06 results.

```

#32:  \int_0^{2 \cdot \pi} \left| \text{SIN}(t) + \text{SIN}\left[2 \cdot t - \frac{\pi}{2}\right] \right| dt
#33:  0
#34:  \int_0^{2 \cdot \pi} \left| \text{SIN}\left[t + \frac{\pi}{6}\right] - \text{COS}\left[2 \cdot t + \frac{\pi}{3}\right] \right| dt
#35:  \frac{3 \cdot \sqrt{3}}{2}
  
```

Hi Rick,

I am quite sure that  $3 \cdot \sqrt{3}$  is correct.

See my TI-Nspire screen below. TI-Nspire is not able to find the exact value for the integral of  $\text{abs}(\sin(t) \dots)$ ,

But it can find the exact value for the shifted and split integrals.

The numerical values confirm the correct result of  $3 \cdot \sqrt{3}$ .

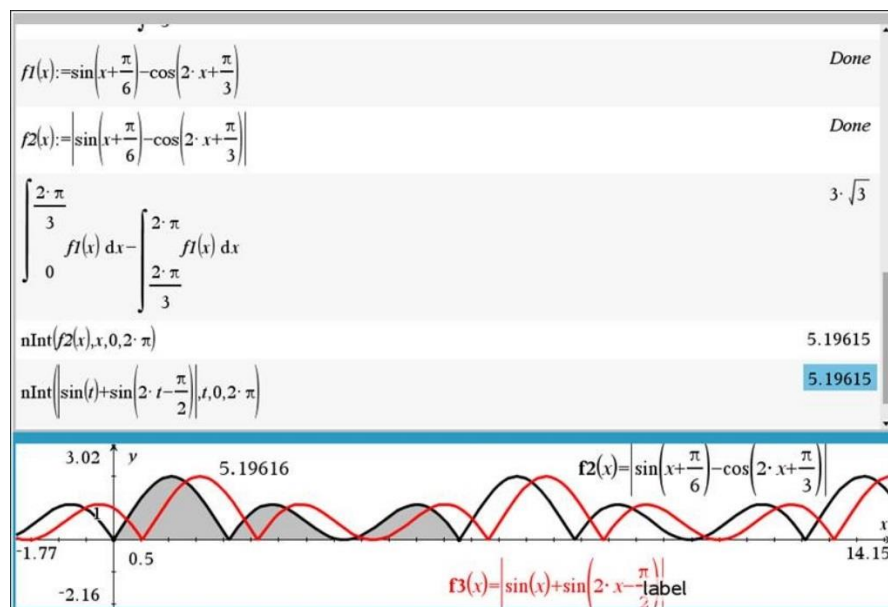
Best regards

Josef

Sometimes I use the DOSBox, too. It's nice to reanimate good old DERIVE.

I like the black screen.

My first version was 1.60!!



$$\#2: \int_0^{2 \cdot \pi} \left| \text{SIN}(t) + \text{SIN}\left(2 \cdot t - \frac{\pi}{2}\right) \right| dt = 3 \cdot \sqrt{3}$$

stepwise simplification:

$$\#3: \int_0^{2 \cdot \pi} \left| \text{SIN}(t) + \text{SIN}\left(2 \cdot t - \frac{\pi}{2}\right) \right| dt$$

$$\text{SIN}(z) \rightarrow -\text{SIN}(z + \pi)$$

$$\#4: \int_0^{2 \cdot \pi} \left| \text{SIN}(t) - \text{SIN}\left(2 \cdot t + \frac{\pi}{2}\right) \right| dt$$

$$\text{SIN}\left(z + \frac{\pi}{2}\right) \rightarrow \text{COS}(z)$$

$$\#5: \int_0^{2 \cdot \pi} |\text{SIN}(t) - \text{COS}(2 \cdot t)| dt$$

## Mail from our Australian friend Steve Arnold:

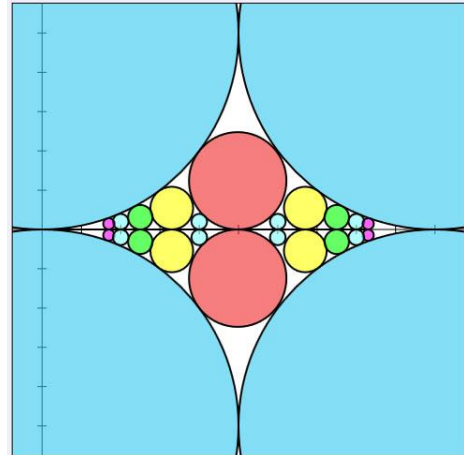
I did one a couple of weeks ago - you may want to have a play with my latest project:

<https://compasstech.com.au/gx/index.html>

One I am particularly happy with is my exploration of Farey Numbers and Ford ("Kissing") Circles:

<https://compasstech.com.au/fareygx/index.html>

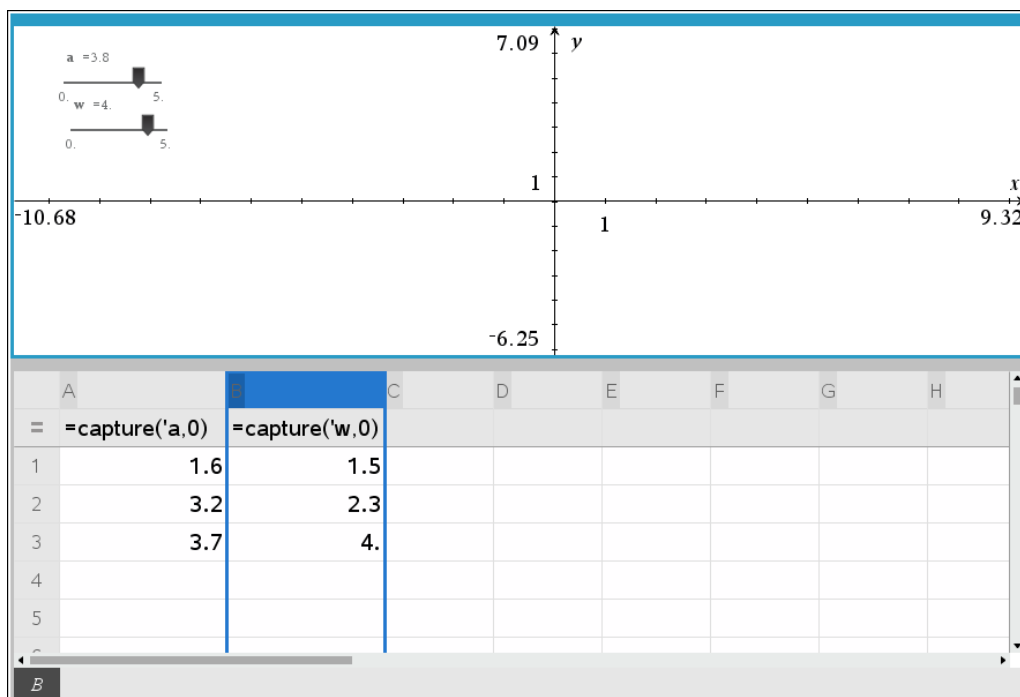
Use the link, enjoy and admire, Josef



## Transferring data from the graphics screen to the spreadsheet on TI-Nspire

When I wanted to reproduce Karl-Heinz' experiments I wanted to port the maximum amplitudes of the vibrations to the Lists & Spreadsheet application applying any keypress. I remembered that there was a Ctrl+D (???) combination with the TI-92/V200 but I could not find how to do with TI-Nspire. So, I asked some friends – and very soon I received answers:

**Hubert Langlotz** gave the advice to use columns in the spreadsheet `capture(variable,0)` and then the combination Ctrl + . transfers the values from the Graphs-app into the respective columns of the spreadsheet. Shame on me that I didn't find this in the manual – and that I didn't remember...



Sebastian’s tool to capture the path of a point:

reset\_appl\_new.tns

Liebe T3ler,

ich habe etwas nützliches programmiert, in Basic. Sozusagen capture 2.0 mit Resetknopf. Ich habe einen Screenshot eines alten Problems von Bärbel gefunden. Dabei ging es um die Basis eines Rechtecks und seine Fläche bzw. den Umfang, die gegeneinander aufgetragen werden.

In dieser Datei kann man den dicken, blauen Punkt bewegen. Im 1x1 Quadrat bildet sich ein achsparalleles Rechteck mit dem Punkt als rechter, oberer Punkt und 0/0 als unterer, linker Punkt.

Ich speichere die Koordinaten des blauen Punktes als px und py und übergebe die in einem Notes-Dokument an das Programm, zusammen mit der reset Variablen.

Das Programm selbst ist hier maximal einfach. Erst wird abgefragt, ob die reset Variable auf 1 steht, wenn das so ist, werden die beiden Listen gelöscht **UND reset wieder auf 0 gesetzt.**

Ist reset=1 werden die aktuellen Werte von px und py in der Liste der x und y -Werte (xx,yy) gespeichert.

Das Programm selbst wird in einer Notesumgebung aufgerufen. Das passiert immer dann, wenn sich eines der Argumente ändert.

Dear T3lers,

I programmed something useful in Basic. So to speak capture 2.0 with reset button.

I found a screenshot of an old problem of Bärbel. It was about the base of a rectangle and its area or perimeter, which are plotted against each other.

In this file you can move the thick blue dot. In the 1x1 square an axis-parallel rectangle is formed with the point as the right, upper point and 0/0 as the lower, left point.

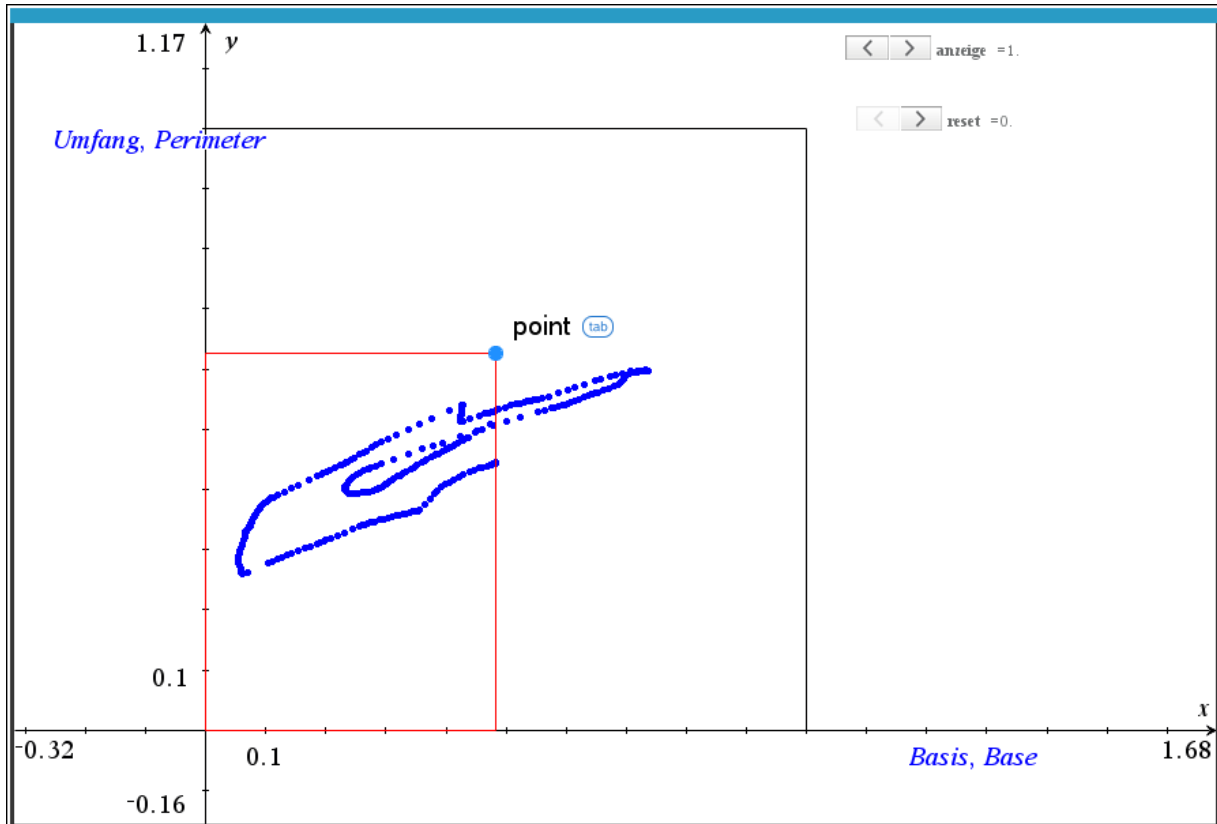
I save the coordinates of the blue point as px and py and pass them in a Notes document to the program, together with the reset variable.

The program itself is maximally simple here. First I ask if the reset variable is set to 1, if so, the two lists are cleared **AND reset is set to 0 again.**

If reset=1, the current values of px and py are stored in the list of x and y values (xx,yy).

The program itself is called in a Notes environment. This happens whenever one of the arguments changes.

You can grab the point and move it around within the square. The slider **anzeige** is used to switch between different presentations of the axes: perimeter vs base, area vs base and area vs perimeter.



A	xx	B	yy	C	umfang	D	flaeche	E	rx	F	ry	G	H	I
=					$= (2 * rx + 2 * ry)$		$= rx * ry$		$= \text{iffn}(xx < 0)$		$= \text{iffn}(yy < 0)$			
1	0.473539	0.796173	0.507885	0.377019	0.473539	0.796173								
2	0.459934	0.789383	0.499727	0.363064	0.459934	0.789383								
3	0.448596	0.78712	0.494286	0.353099	0.448596	0.78712								
4	0.437258	0.78033	0.487035	0.341206	0.437258	0.78033								
5	0.434991	0.78033	0.486128	0.339436	0.434991	0.78033								
6	0.428188	0.778066	0.482502	0.332150	0.428188	0.778066								
A7	=0.4735393537836													

main(reset,px,py)	Done
ax:=rx[dim(rx)]	0.48261
ay:=ry[dim(ry)]	0.626421
aax:=ax	0.48261
aay:=ay	0.626421

main	1/7
Define main(rs,ppx,ppy)=	
Prgm	
If rs=1 Then	
reset:=0	
xx:={ }	
yy:={ }	
EndIf	
xx:=augment(xx,{ppx})	
yy:=augment(yy,{ppy})	
EndPrgm	

Just move the point and notice its path in the spreadsheet.

Reset\_simple\_new.tns

The screenshot shows a TI-Nspire interface. On the left, a coordinate plane with x and y axes ranging from -10 to 10. A point is labeled "point (xx,yy)" with a blue dot at coordinates (3.15, 1.61). A "Reset Button" with left and right arrows is visible. On the right, a spreadsheet window shows a table with columns A (xx) and B (yy). The first row contains the values 3.14943 and 1.61104. The spreadsheet title is "track" and the page number is 1/7.

```

Define track(rs)=
Prgm
If rs=1 Then
  xx:={ }
  yy:={ }
  reset:=0
EndIf
xx:=augment(xx,{px})
yy:=augment(yy,{py})
EndPrgm
  
```

	A xx	B yy	C	D
=				
1	3.14943	1.61104		
2				
3				
4				
5				

The screenshot shows the same TI-Nspire interface as above, but the point has moved to (2.05, 1.38). The spreadsheet now contains five rows of data, showing the path of the point as it moves. The spreadsheet title is "track" and the page number is 1/7.

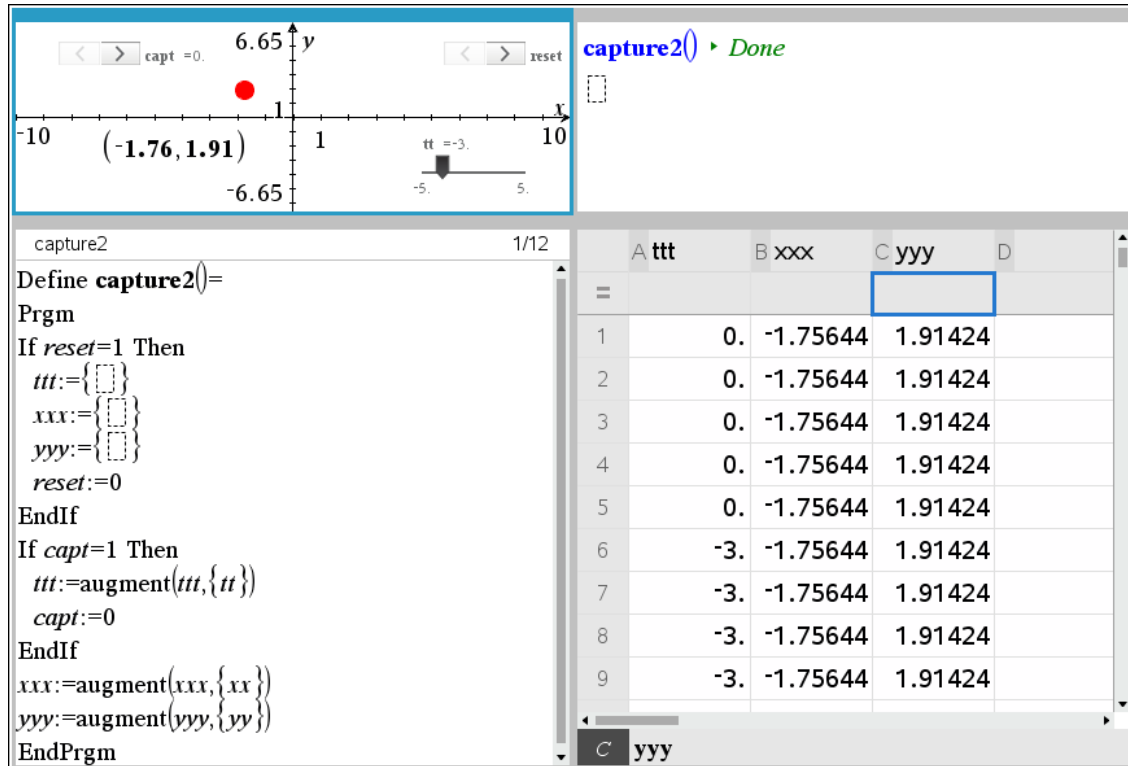
```

Define track(rs)=
Prgm
If rs=1 Then
  xx:={ }
  yy:={ }
  reset:=0
EndIf
xx:=augment(xx,{px})
yy:=augment(yy,{py})
EndPrgm
  
```

	A xx	B yy	C	D
=				
1	3.14943	1.61104		
2	2.96552	1.56434		
3	2.87356	1.51764		
4	2.78161	1.51764		
5	2.73563	1.47095		

This is a presentation of several ways to capture the coordinates of the point (slider **capt**, slider **tt** and moving the point) in one graphs-application.

Capture2\_v2.tns



The screenshot shows a TI-Nspire application interface. At the top, there is a coordinate plane with a red point at (-1.76, 1.91). A slider for 'tt' is set to -3. Below the graph is a code editor for a program named 'capture2'. The code is as follows:

```

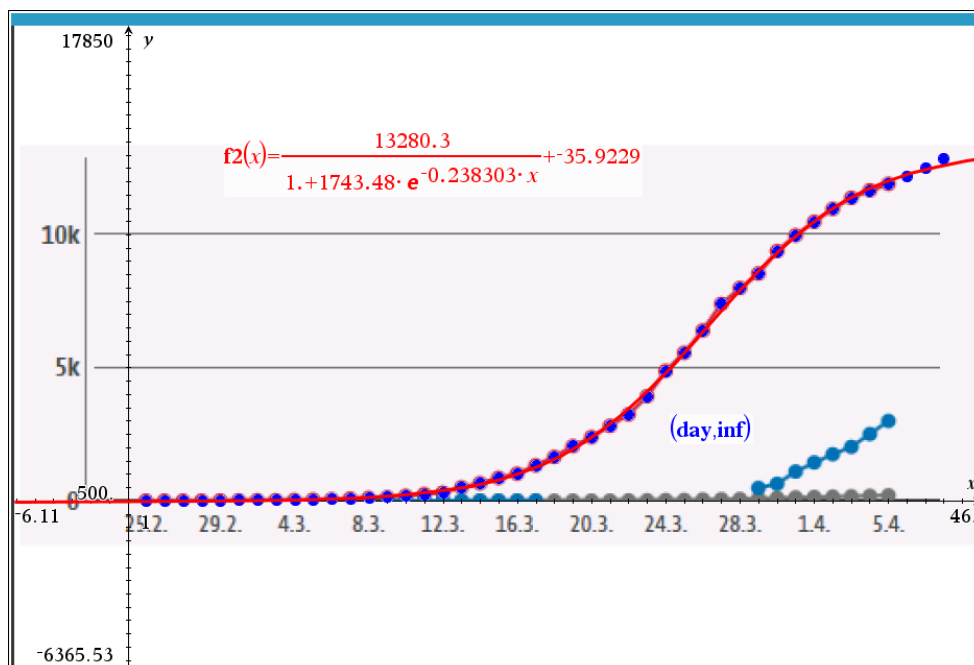
Define capture2()=
Prgm
If reset=1 Then
  ttt:={ }
  xxx:={ }
  yyy:={ }
  reset:=0
EndIf
If capt=1 Then
  ttt:=augment(ttt,{tt})
  capt:=0
EndIf
xxx:=augment(xxx,{xx})
yyy:=augment(yyy,{yy})
EndPrgm

```

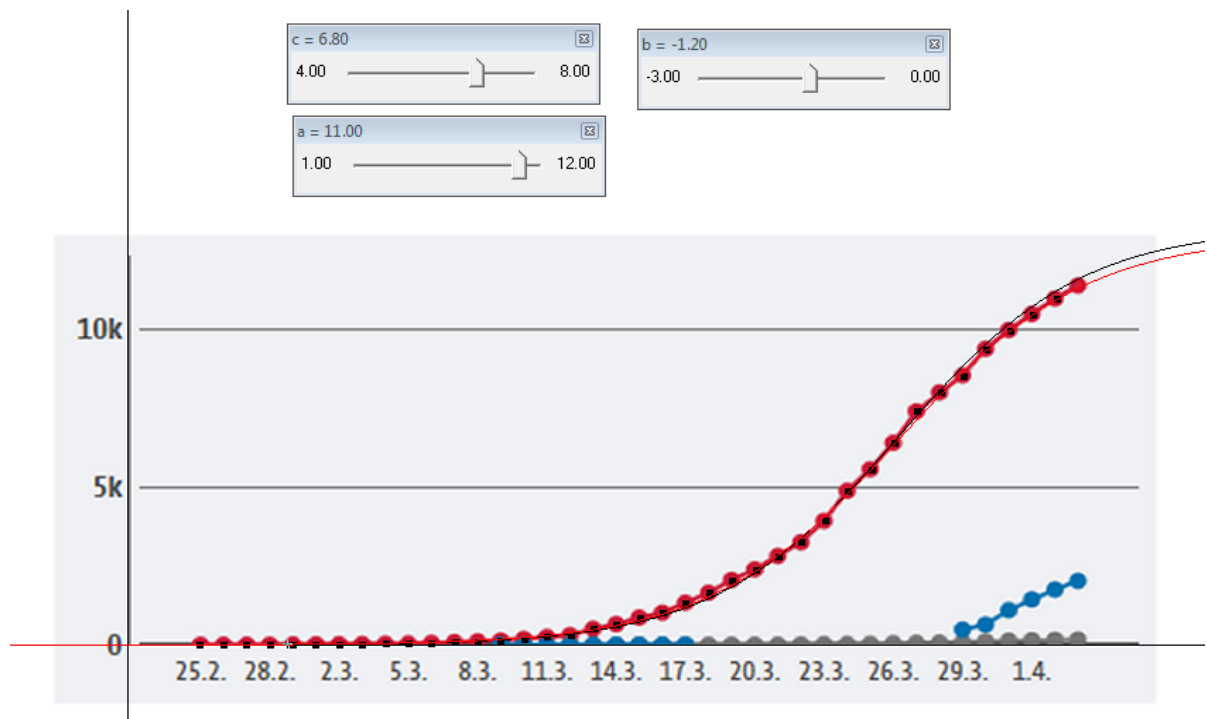
To the right of the code editor is a data table with columns 't', 'xxx', and 'yyy'. The table contains the following data:

	t	xxx	yyy
1	0.	-1.75644	1.91424
2	0.	-1.75644	1.91424
3	0.	-1.75644	1.91424
4	0.	-1.75644	1.91424
5	0.	-1.75644	1.91424
6	-3.	-1.75644	1.91424
7	-3.	-1.75644	1.91424
8	-3.	-1.75644	1.91424
9	-3.	-1.75644	1.91424

**Modelling the spread of the Corona Virus in Austria** from 25 February until begin of April. With the diagram from ORF (Austrian Broadcasting Corporation) as background and Logistic regression.







Same with DERIVE (background picture) + data points + regression line (calculated and slider bar generated). Logistic Regression from Don Phillips, DNL#79.

## Simulation of the Corona Epidemic

Sebastian Rauh

German version above

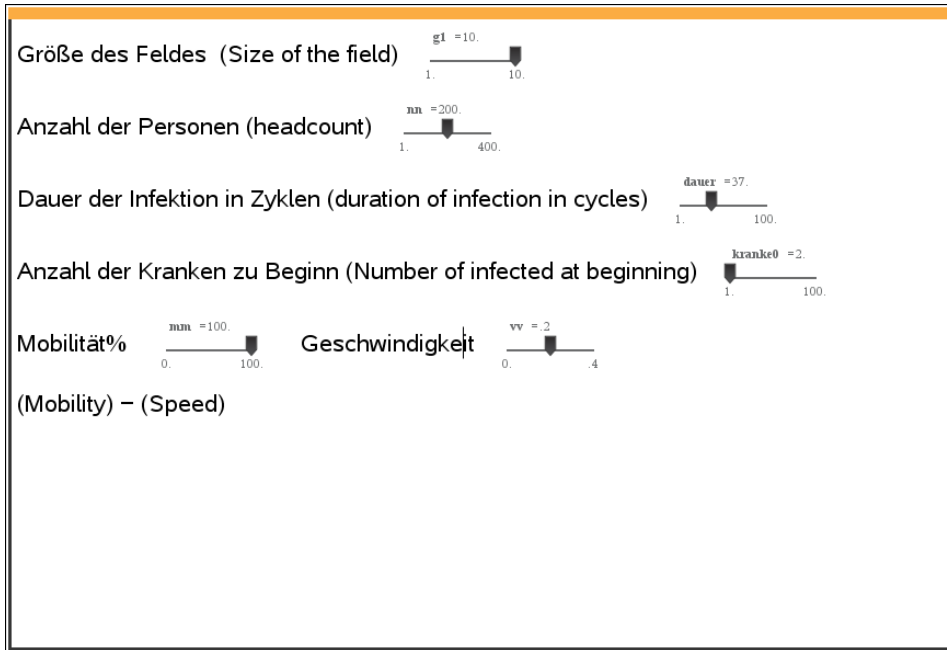
**Abstract**  
 An epidemic is to be simulated here. People are walking around in a square, they cannot leave the square. The persons can be healthy (blue), sick (red) or immune (green).  
 A healthy person is infected when it meets a sick person. A sick person is contagious for a certain period of time, after which they automatically become immune.

**Overview**  
 On the 3rd page you can see a graphic representation of the simulation, the persons are in the square and cannot leave it.  
 On the top right is a calculator, there you can reinitialize the simulation, on the bottom right is a graph showing how many people are healthy, sick and immune.

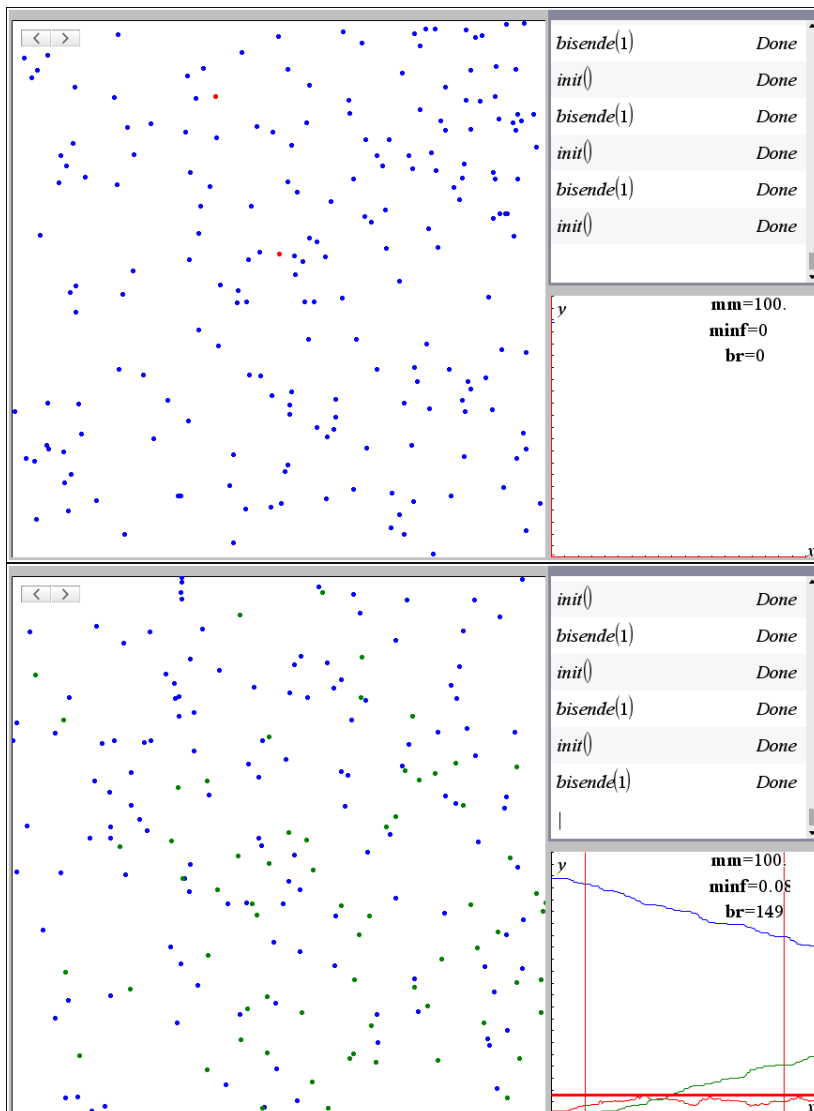
**Instructions**  
 To (re)start the simulation, call the initialization program on the 3rd page top right `init()`  
 The simulation is started via the slider on the graph page of the 3rd page, each movement generates a step which is also displayed graphically.

The complete English and German instruction is given in the Notes of the InfektV1.6.tns file.

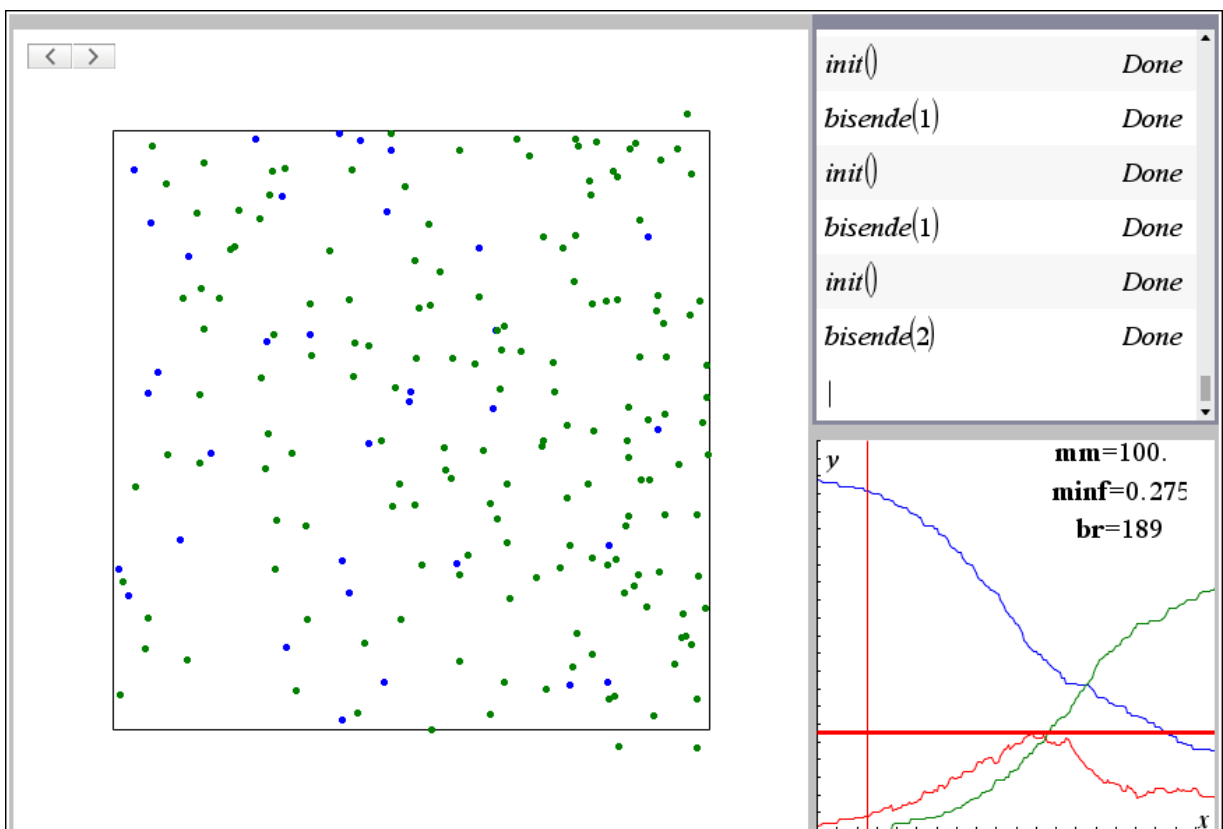
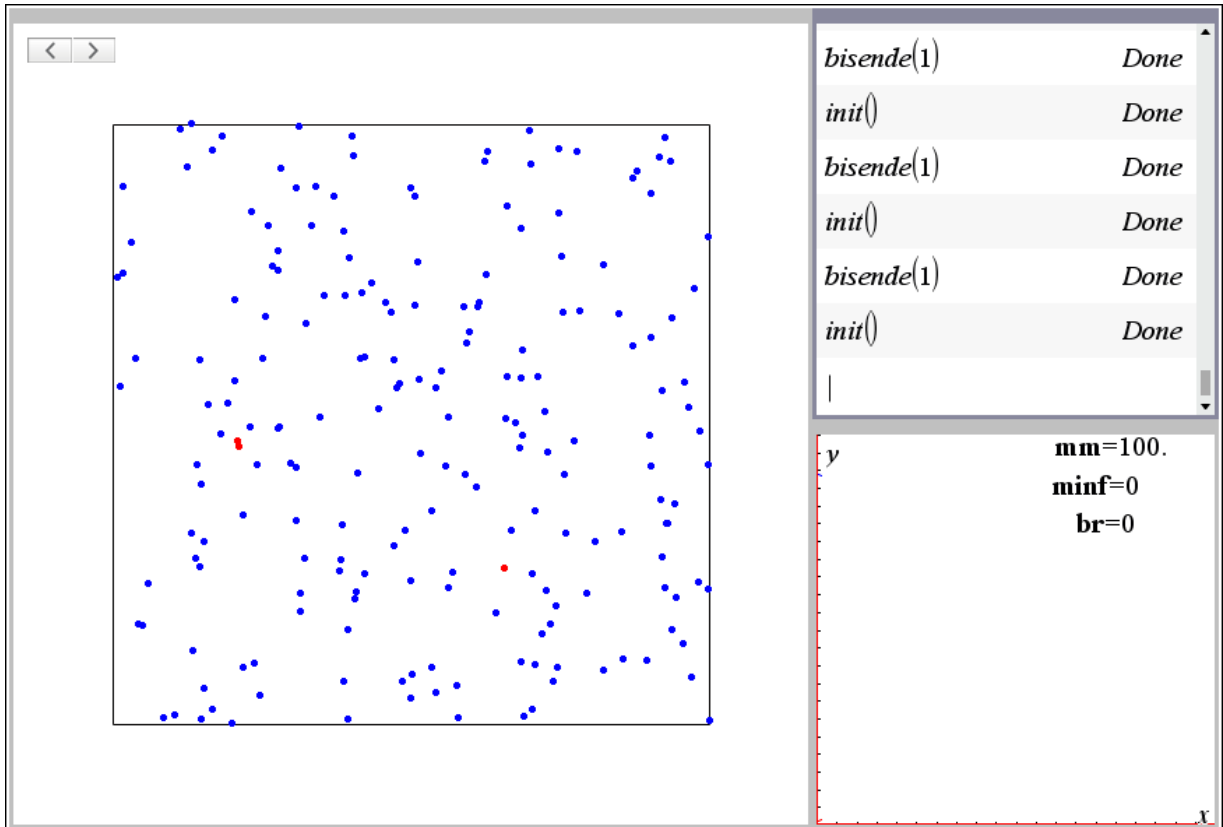
[Die vollständige deutsche und englische Erläuterung findet sich in den Notes der Datei InfektV1.6.tns](#)



See the two red points (infected persons) in the start picture followed by the final graph.



Here is a second simulation. `bisende(1)` runs one simulation, `bisende(n)` runs  $n$  simulations with the same parameters remaining and stores the resulting data in the spreadsheet. The slider can be used to perform the simulation step by step. ([bisende = until end](#))



## MathTools from agm to zetaprim

```

F1 Control F2 I/O F3 Var F4 Find... F5 Mode
:agm(a,b)
:Func
:@AGM(a,b) returns the arithmetic-geometric mean. It returns {a[N],N}+@Bhuvanesh Bhatt
:Local q,tmp
:If not mathtool\iscmplxn(a) or not mathtool\iscmplxn(b):Return "Error: invalid argument(s)"
:{a,b}→q:0→a
:While abs(q[1]-q[2])>1.e-12
:1/2*(q[1]+q[2])→tmp:(1/2*(q[1]+q[2]),√(q[1]*q[2]))→q:a+1→a
:EndWhile
:{approx(q[1]),a}
:EndFunc
MATHTOOL RAD EXACT FUNC

```

```

F1 Control F2 I/O F3 Var F4 Find... F5 Mode
:Bhatt
:Local q,tmp
:If not mathtool\iscmplxn(a) or not mathtool\iscmplxn(b):Return "Error: invalid argument(s)"
:{a,b}→q:0→a
:While abs(q[1]-q[2])>1.e-12
:1/2*(q[1]+q[2])→tmp:(1/2*(q[1]+q[2]),√(q[1]*q[2]))→q:a+1→a
:EndWhile
:{approx(q[1]),a}
:EndFunc
MATHTOOL RAD EXACT FUNC

```

<pre> agm(12,7,10) {9.5,9.16515138991,1.} {9.33257569496,9.33107379695,2.} {9.33182474595,9.33182471574,3.} {9.33182473085,9.33182473085,4.} {9.33182473085,9.33182473085,4.} © Disp command inactive agm(12,7,10) {9.33182473085,9.33182473085,4.} agm(27,7,3) {15.3307423465,15.3306817023,3.} agm(17,a,3) "Error: invalid argument(s)"   </pre>	<pre> "agm" stored successfully Define LibPub agm(x,y,e)= Func © Bhuvanesh Bhatt Local a,g,n,q q:={x,y}:n:=1 If not mathtool\iscmplxn(x) or not mathtool\iscmplxn(y) While  q[1]-q[2] &gt;10^-e a:=1/2*(q[1]+q[2]) g:=√(q[1]*q[2]) q:={a,g} © Disp approx({a,g,n}) n:=n+1 EndWhile approx({approx(a),approx(g),n-1}) EndFunc </pre>
--	---

DERIVE Version:

```

agm(x, y, e, a, g, n, q) :=
Prog
If ~ NUMBER?(x) v ~ NUMBER?(y)
RETURN "invalid arguments"
q := [x, y]
n := 1
#1: Loop
If ABS(q[1] - q[2]) < 10^(-e) exit
a := AVERAGE(q[1], q[2])
g := √(q[1]*q[2])
q := [a, g]
n := n + 1
[APPROX(a), n - 1]
#2: agm(27, 7, 3) = [15.33074234, 3]
#3: agm(12, 7, 10) = [9.331824730, 4]
#4: agm(3 + i, 2 - 7*i, 4) = [3.368064117 - 2.671659005*i, 4]
#5: agm(a, 7, 10) = invalid arguments

```

Steve Arnold pointed out that there is also a harmonic-geometric mean. (see next DNL!)

[https://en.wikipedia.org/wiki/Arithmetic%E2%80%93geometric\\_mean](https://en.wikipedia.org/wiki/Arithmetic%E2%80%93geometric_mean)