

**THE DERIVE - NEWSLETTER #126**

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**THE BULLETIN OF THE**

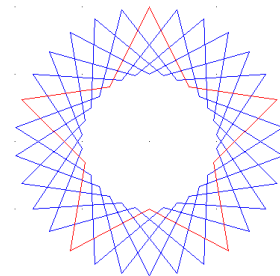


**USER GROUP**

**C o n t e n t s :**

**+ CAS-TI**

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**June 2022**

z-lib.org is a rich resource for downloading complete textbooks free of charge

<https://z-lib.org/>

Here are some examples:

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*Tristan Needham*, Visual Complex Analysis:

<https://de.1lib.at/book/974187/196adc?dsource=recommend>

*L. L. Pennisi*, Elements of Complex Variables, <https://de.1lib.at/book/5231569/bd1540>

Digitale Bildverarbeitung: *Martin Werner* <https://de.1lib.at/book/17130018/b1be8a>,

*Bernd Jähne* <https://de.1lib.at/book/542128/9c43f8>

*Alasdair McAndrew*, A Computational Introduction to Digital Image Processing:

<https://de.1lib.at/book/2950021/671c52>

*Tinku Acharya & Ajoyk. Ray*, Image Processing: Principles and Applications:

<https://de.1lib.at/book/437632/283085?dsource=recommend>

*Sidney a. Morris*, Topology without Tears,

<https://www.topologywithouttears.net/topbook.pdf>

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**Neue interaktive Online-Aufgaben zur spielerischen Förderung und Diagnose des räumlichen Denkens**

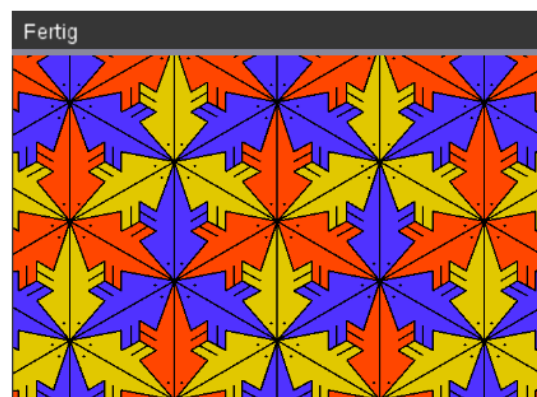
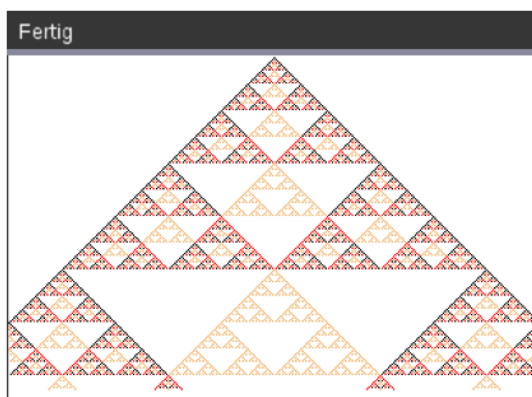
**Ideal for the weeks leading up to the end of school:**

**New interactive online tasks for playful promotion and diagnosis of spatial thinking.**

Webseite mit 1.039 interaktiven Aufgaben > [https://geometriedidaktik.at/?page\\_id=1882](https://geometriedidaktik.at/?page_id=1882)

[https://resources.t3europe.eu/t3europe-home?resource\\_id=3576&cHash=f435babab7632e77c7ee7820c8fd953d](https://resources.t3europe.eu/t3europe-home?resource_id=3576&cHash=f435babab7632e77c7ee7820c8fd953d)

[https://resources.t3europe.eu/t3europe-home?resource\\_id=3572&cHash=3fc404820251f947f687e70aae96aafa](https://resources.t3europe.eu/t3europe-home?resource_id=3572&cHash=3fc404820251f947f687e70aae96aafa)



Dear DUG-Members,

I greet you from the summery Würmla, Lower Austria. Before the summer vacations want to deliver the DNL126. Thanks to your contributions it is not difficult to fill the DNL again and again. In addition, the inclusion in our journal often results in further interesting communication with the authors.

A good example can be found here in connection with the "Cello Tangents" by Robert Haas (see our User Form pp 38). There is also a regular exchange of e-mails with Wolfgang Alvermann and Michel Beaudin.

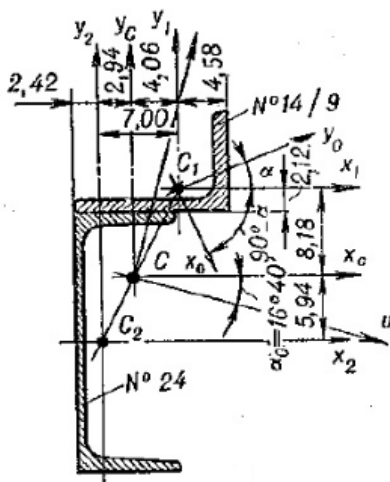
In the article of Felipe Martinez Vargas, you will find some topics that have been covered in earlier DNLs. However, it will be continued in the next DNL and leads to exciting applications such as the realization of technical drawings with *DERIVE* (below left a construction drawing and below right its *DERIVE* execution).

My special note today points to the links for downloading textbooks. With the website z-lib.org I have discovered a true treasure trove. Since I am now a little busy with function theory - last time during my studies in 1966 - I have put relevant works first.

Note also the links to two great webinars. All related files and the recordings of the presentations can be obtained (Binomial Theorem and Maurits Escher).

I wish you a nice summer and remain until next time

Yours Josef



Liebe DUG-Mitglieder,

Ich grüße Euch aus dem sommerlichen Würmla, Niederösterreich. Vor den Sommerferien will ich noch den DNL126 abliefern. Dank Eurer Beiträge ist es nicht schwer, den DNL immer wieder zu füllen. Darüber hinaus ergibt sich bei der Aufnahme in unser Journal oft eine weitere interessante Kommunikation mit den Autoren.

Ein gutes Beispiel findet sich hier im Zusammenhang mit den „Cello Tangenten“ von Robert Haas. So gibt es auch immer wieder mit Wolfgang Alvermann und Michel Beaudin einen regelmäßigen Austausch von e-mails.

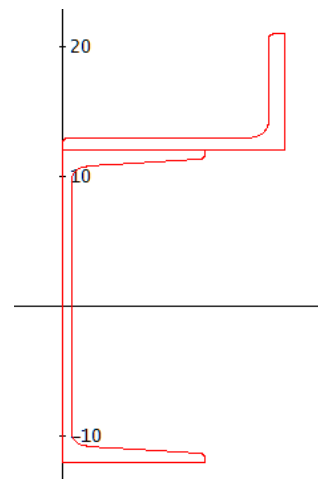
Im Beitrag von Felipe Martinez Vargas gibt es zum Teil schon früher behandelte Themen. Er wird aber im nächsten DNL fortgesetzt und führt zu spannenden Anwendungen wie zur Realisierung technischer Zeichnungen mit *DERIVE* (links unten die Konstruktionszeichnung und rechts unten die *DERIVE*-Ausführung).

Mein besonderer Hinweis gilt heute den Links zum Download von Fachbüchern. Mit der Webseite z-lib.org habe ich eine wahre Fundgrube entdeckt. Da ich mich jetzt ein wenig mit Funktionentheorie beschäftige - zuletzt während des Studiums 1966 – habe ich diesbezügliche Werke vorangestellt.

Beachtet auch die Links zu zwei sehr schönen Webinaren. Alle zugehörigen Dateien und die Aufzeichnungen der Präsentationen können bezogen werden (Binomischer Ls., M. Escher).

Ich wünsche Euch einen schönen Sommer und verbleibe bis zum nächsten Mal

Euer Josef



The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & CAS-TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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### Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & CAS-TI Newsletter* will be.

Next issue:

September 2022

### **Preview: Contributions waiting to be published**

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER  
Wonderful World of Pedal Curves, J. Böhm, AUT  
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT  
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT  
Logos of Companies as an Inspiration for Math Teaching  
Exciting Surfaces in the FAZ, BooleanPlots.mth, P. Schofield, UK  
Old traditional examples for a CAS – What's new? J. Böhm, AUT  
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ  
Tutorials for the NSpireCAS, G. Herweyers, BEL  
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA  
Another Approach to Taylor Series, D. Oertel, GER  
Charge in a Magnetic Field, H. Ludwig, GER  
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover  
Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT  
A Collection of Special Problems, Hemme's problems, W. Alvermann, GER  
DERIVE Bugs? D. Welz, GER  
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm, AUT  
The Gap between Poor and Rich, J. Böhm, AUT  
More functions from M. Myers and from Bhuvanesh's Mathtools-library  
TaxiCab Conics, R. Haas, USA  
QR-Code light, Random numbers following a given distribution  
Polygons and their properties, Quartiles, F. de Jesús Martínez Vargas, Mexico  
Penalty Shootout mathematics, B. Grabinger, GER  
Sparse Matrices, 153 is another Special Number, and others

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### Résumé 1 – Part 5

## Résolution symbolique, résolution numérique d'équations et de systèmes d'équations Rappels et compléments en équations différentielles

### 3- Transformées de Laplace : rappels et compléments

Il y a plusieurs bonnes raisons d'utiliser la transformée de Laplace et l'exemple suivant fournit une réponse à cette question.

**4.1 Exemple** Un réservoir contient initialement 500 L d'une solution saline de concentration 2 kg/L. Pendant les 10 premières minutes, on ouvre une valve A, ce qui a pour effet d'injecter 12 L/min de solution saline de concentration 4 kg/L. Après 10 minutes, c'est la valve B qui apporte 6 kg/L de concentration à un taux de 12 L/min. Une valve de sortie C retire 12 L/min, gardant ainsi le volume constant. Il faut trouver la concentration de sel dans le réservoir en fonction du temps.

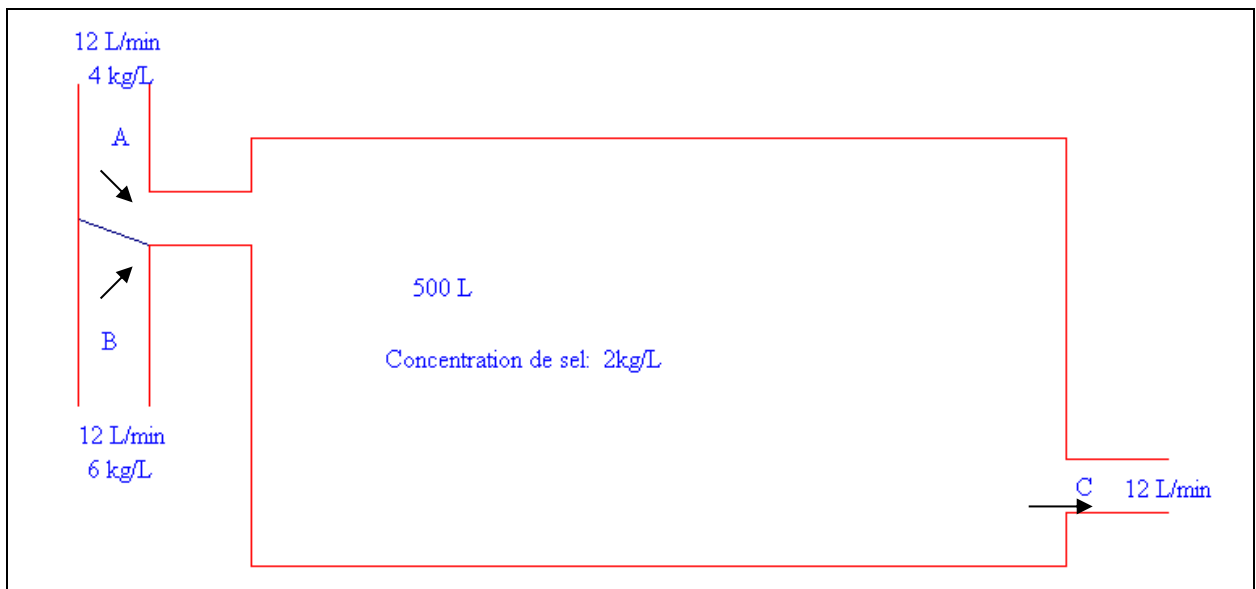


Figure 4.1

$$\text{La réponse est } \begin{cases} 4 - 2e^{-3t/125} & \text{si } t < 10 \\ 6 - 2e^{-3t/125} - 2e^{-3(t-10)/125} & \text{si } t \geq 10 \end{cases}$$

Voici quelques bonnes raisons (il y en a d'autres) d'utiliser la transformée de Laplace :

Cela évite, lorsqu'on résout des équations différentielles avec membre de droite (« input ») continu par morceaux, d'effectuer plusieurs intégrales et d'ajuster à chaque fois les constantes d'intégration. D'un point de vue de l'analyse, la transformation de Laplace réalise une correspondance entre espaces fonctionnels et d'un point de vue opérationnel, on passe d'une équation différentielle à une équation algébrique. On peut introduire des fonctions « impulsions » pour lesquelles les techniques de coefficients indéterminés et/ou de variation des paramètres sont inefficaces.

Rappelons donc certaines définitions, certaines propriétés et prouvons certains résultats non démontrés lors d'un cours d'équations différentielles du premier cycle.

**4.2 Définitions** Soit  $f$  une fonction définie pour  $t \geq 0$ . On dit que cette fonction est d'ordre exponentiel pour  $t \rightarrow \infty$  s'il existe des constantes non-négatives  $M, c$  et  $T$  satisfaisant  $|f(t)| \leq Me^{ct}$  si  $t \geq T$ . Alors si l'intégrale impropre

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

converge pour certaines valeurs de  $s$ , son résultat est une fonction appelée la transformée de Laplace de  $f$ . L'opérateur linéaire qui associe la fonction  $f$  avec la fonction  $F$  est appelé transformation de Laplace et est dénoté ici par  $\mathcal{L}$  :

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)] \quad (\alpha, \beta \in \mathbb{R})$$

On dénote habituellement la correspondance par  $f(t) \leftrightarrow F(s)$ . La transformée de Laplace inverse est obtenue par résolution de l'équation intégrale (nous verrons comment les variables complexes nous permettent d'obtenir ce résultat). Une notation commune des livres de mathématiques est d'utiliser  $u(t)$  pour désigner la fonction échelon-unité (ou fonction de Heaviside), à savoir

$$u(t) = \frac{1 + \text{sign}(t)}{2} = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

On trouve donc que  $\mathcal{L}[u(t)] = \frac{1}{s}$  ( $s > 0$ ).

**4.3 Exemple** « Mini table » de transformées de Laplace. Pour chacune des fonctions  $f$  suivantes (colonne de droite dans le tableau ci-bas), l'intégrale impropre (19) converge si l'on choisit  $s > a$  et on trouve le  $F(s)$  indiqué dans la colonne de gauche : en utilisant la factorisation dans  $\mathbf{R}$  et la technique des fractions partielles, nous n'avons pas besoin d'une « table » de transformées de Laplace trop volumineuse, celle de la figure 4.2 est très utile par exemple. Si des facteurs quadratiques irréductibles répétés sont présents, la convolution pourra nous servir (voir plus loin).

$F(s)$	$f(t)$
$\frac{1}{(s-a)^k} \quad (k > 0)$	$\frac{t^{k-1}e^{at}}{\Gamma(k)} \quad \left( = \frac{t^{k-1}e^{at}}{(k-1)!} \text{ si } k \in \mathbf{N}^* \right)$
$\frac{1}{(s-a)^2 + \omega^2}$	$\frac{1}{\omega} e^{at} \sin(\omega t)$
$\frac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos(\omega t)$

**Figure 4.2**

Par abus de notation, on écrit souvent  $F(s) = \mathcal{L}[f(t)]$  ou encore  $f(t) = \mathcal{L}^{-1}[F(s)]$  pour indiquer la correspondance  $f(t) \leftrightarrow F(s)$ . Voici une [table](#) plus complète de transformées de Laplace.

**4.4 Remarque** La fonction gamma  $\Gamma$  est définie par  $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$  et cette intégrale impropre converge pour  $x > 0$ . Cette fonction satisfait notamment aux propriétés suivantes:

$$\Gamma(1) = 1, \Gamma(n) = (n-1)! \text{ si } n \in \mathbb{N}^*, \Gamma(x+1) = x\Gamma(x), \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

**4.5 Définitions** On dit qu'une fonction  $f$  est continue par morceaux sur un intervalle  $[a, b]$  si l'on peut subdiviser l'intervalle en un nombre fini de sous-intervalles tels que  $f$  est continue sur chacun des sous-intervalles ouverts et si  $f$  possède des limites en chacune des extrémités de ces sous-intervalles (limite à droite (resp. à gauche) pour l'extrémité gauche (resp. droite)). On dit que  $f$  est continue par morceaux sur  $[0, \infty[$  si  $f$  est continue par morceaux sur chaque sous-intervalle borné de  $[0, \infty[$ . Le saut de  $f$  au point  $c$  est noté  $j_f(c)$  et défini par

$$j_f(c) = f(c+) - f(c-) \text{ où } f(c+) = \lim_{\varepsilon \rightarrow 0^+} f(c + \varepsilon) \text{ et } f(c-) = \lim_{\varepsilon \rightarrow 0^+} f(c - \varepsilon).$$

**4.6 Théorème a) Condition suffisante (mais non nécessaire) pour l'existence de la transformée de Laplace** Si  $f$  est continue par morceaux pour  $t \geq 0$  et d'ordre exponentiel pour  $t \rightarrow \infty$ , alors la transformée de Laplace de  $f$  existe pour  $s > c$ . De plus,  $\lim_{s \rightarrow \infty} F(s) = 0$ .

**b) Unicité de la transformée inverse « presque partout »** Si  $f$  et  $g$  satisfont les hypothèses en a) et si  $F(s) = G(s)$  pour tout  $s > c$ , alors  $f(t) = g(t)$  partout sur  $[0, \infty[$  où  $f$  et  $g$  sont continues.

**c) Transformée de la dérivée** Si  $f$  est continue, dérivable par morceaux pour  $t \geq 0$  et d'ordre exponentiel pour  $t \rightarrow \infty$ , alors la transformée de Laplace de  $f'$  existe et on a

$$\mathfrak{L}[f'(t)] = sF(s) - f(0)$$

**d)** Si, en c),  $f$  n'est que continue par morceaux, avec des discontinuités (sauts finis) localisées aux points  $t_1, t_2, t_3, \dots$  et si  $j_f(t_n) = f(t_n+) - f(t_n-)$  est le saut de  $f(t)$  en  $t = t_n$ , alors, en supposant que la transformée de  $f'$  existe, on a

$$\mathfrak{L}[f'(t)] = sF(s) - f(0) - \sum_{n=1}^{\infty} e^{-st_n} j_f(t_n)$$

Preuve : la preuve de b) découlera du théorème d'inversion en analyse complexe. Prouvons a). On a, si  $b > 0$ , que

$$\left| \int_0^b e^{-st} f(t) dt \right| \leq \int_0^b |e^{-st} f(t)| dt \leq \int_0^b |e^{-st} M e^{ct}| dt = M \int_0^b e^{-(s-c)t} dt \leq M \int_0^{\infty} e^{-(s-c)t} dt = \frac{M}{s-c}$$

et donc,  $|F(s)| \leq \int_0^{\infty} |e^{-st} f(t)| dt \leq \frac{M}{s-c} \rightarrow 0$  si  $s \rightarrow \infty$ .

Pour c) et d), montrons que  $\lim_{b \rightarrow \infty} \int_0^b e^{-st} f'(t) dt$  existe et trouvons cette limite. Si  $b$  est fixé, soient  $t_1, t_2, \dots, t_{k-1}$  les points intérieurs à l'intervalle  $[0, b]$  où la fonction  $f$  pourrait être discontinue par sauts. Posons  $t_0 = 0$  et  $t_k = b$ . Alors, on peut utiliser l'intégration par parties sur chacun des intervalles ouverts  $t_{n-1}, t_n$  où  $f'$  est continue et obtenir

$$\begin{aligned} \int_0^b e^{-st} f'(t) dt &= \sum_{n=1}^k \int_{t_{n-1}}^{t_n} e^{-st} f'(t) dt = \sum_{n=1}^k \left( \left[ e^{-st} f(t) \right]_{t_{n-1}}^{t_n} + s \int_{t_{n-1}}^{t_n} e^{-st} f(t) dt \right) \\ &= -f(0) - \sum_{n=1}^{k-1} e^{-st_n} j_f(t_n) + e^{-sb} f(b) + s \int_0^b e^{-st} f(t) dt \end{aligned}$$

où  $j_f(t_k)$  est nul car  $f$  est continue en  $b = t_k$ . De plus, si  $b > c$ , alors  $e^{-sb} f(b) \rightarrow 0$  si  $b \rightarrow \infty$ . Donc le résultat est établi, en laissant  $b$  tendre vers l'infini. ♦

**4.7 Exemple** Illustrons les parties c) et d) du théorème 4.6. Considérons les deux fonctions suivantes :  $f1$  qui est continue partout et dérivable par morceaux et  $f2$  qui est continue par morceaux (et aussi dérivable par morceaux) mais avec un saut de 1 en  $t = 1$ .

$$f1(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases} \quad f2(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2, & t \geq 1 \end{cases}$$

On trouve aisément leur transformée de Laplace respective puisque  $f1(t) = t + (1-t)u(t-1)$  tandis que  $f2(t) = t + (2-t)u(t-1)$  :

$$F1(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} \quad F2(s) = \frac{1}{s^2} + \left( \frac{1}{s} - \frac{1}{s^2} \right) e^{-s}$$

Tandis que les dérivées sont les suivantes et égales (sans utiliser les fonctions généralisées) :

$$f1'(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases} \quad f2'(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

En calculant les transformées de Laplace, puisque  $f1'(t) = f2'(t) = u(t) - u(t-1)$ , on trouve

$$\mathcal{L}[f1'(t)] = \mathcal{L}[f2'(t)] = \frac{1}{s} - \frac{e^{-s}}{s}.$$

Le théorème 4.6 c) est bien satisfait puisque  $\mathcal{L}[f1'(t)] = \frac{1}{s} - \frac{e^{-s}}{s}$  tandis que

$$sF1(s) - f(0) = s \left( \frac{1}{s^2} - \frac{e^{-s}}{s^2} \right) - 0 = \frac{1}{s} - \frac{e^{-s}}{s}.$$

C'est bien identique. Le théorème 4.6 d) est satisfait puisque  $\mathcal{L}[f2'(t)] = \frac{1}{s} - \frac{e^{-s}}{s}$  tandis que

$$sF2(s) - f(0) - e^{-s} \cdot 1 = s \left( \frac{1}{s^2} + \left( \frac{1}{s} - \frac{1}{s^2} \right) e^{-s} \right) - 0 - e^{-s} = \frac{1}{s} - \frac{e^{-s}}{s}.$$



Les étudiants qui maîtrisent bien les fonctions généralisées peuvent les utiliser. Par exemple si l'on prend la fonction  $f(t) = u(t - 1)$ , alors on a  $f'(t) = \delta(t - 1)$  et le théorème 4.6 c) est satisfait puisque  $e^{-s} = s \frac{e^{-s}}{s} - 0$ . Si l'on ne travaille pas avec les fonctions généralisées, le théorème 4.6 d) s'applique : on a alors  $f'(t) = 0$  sauf en  $t = 1$  où la dérivée n'existe pas et le saut en  $t = 1$  est de 1. Et  $0 = s \frac{e^{-s}}{s} - 1 \cdot e^{-s}$ .

**4.8 Autres propriétés importantes** Du théorème 4.6 c), on tire, si  $f(0) = 0$ , que  $f'(t) \leftrightarrow sF(s)$ . Ainsi, « la dérivation dans le domaine de  $t$  correspond à la multiplication dans le domaine de  $s$  ». On a la propriété « duale » : l'intégration dans le domaine du  $t$  correspond à la division dans le domaine du  $s$  :

$$\mathcal{L} \left[ \int_0^t f(\tau) d\tau \right] = \frac{F(s)}{s} .$$

De plus, les propriétés de translation et de convolution sont fréquemment utilisées ; dans ce qui suit,  $u(t)$  désigne la fonction échelon-unité de Heaviside et  $x(t) \leftrightarrow X(s)$ ,  $h(t) \leftrightarrow H(s)$  :

$e^{at} x(t)$	$X(s-a)$
$x(t-a)u(t-a)$	$e^{-as} X(s)$
$\int_0^t x(\tau)h(t-\tau) d\tau$	$X(s)H(s)$

Figure 4.3

C'est l'intégrale qu'on voit dans le tableau précédent de la figure 4.3 et qui est appelée la « convolution des fonctions  $x$  et  $h$  » et qu'on dénote souvent par  $x(t) * h(t)$ . Voici une preuve de la propriété de convolution. Remarquons au départ que la convolution est commutative puisque, par changement de variables, on peut écrire

$$f(t) * g(t) \equiv \int_0^t f(\tau)g(t-\tau) d\tau = \int_{w=t-\tau}^0 f(t-w)g(w)(-dw) = \int_0^t g(w)f(t-w)dw = g(t) * f(t).$$

On a, en utilisant la définition de transformée de Laplace et la propriété du milieu de la table ci-haut (avec  $a$  remplacé par  $\tau$ )

$$F(s)G(s) = F(s) \int_0^\infty g(\tau)e^{-s\tau} d\tau = \int_0^\infty g(\tau)F(s)e^{-s\tau} d\tau = \int_0^\infty g(\tau)\mathcal{L}[f(t-\tau)u(t-\tau)] d\tau = \int_0^\infty g(\tau) \int_0^\infty f(t-\tau)u(t-\tau)e^{-st} dt d\tau = \int_0^\infty g(\tau) \int_\tau^\infty f(t-\tau)e^{-st} dt d\tau$$

où l'on a utilisé, dans la dernière égalité, le fait que  $u(t-\tau) = 0$  si  $t < \tau$ . Si l'on dessine le domaine d'intégration de la dernière intégrale double, nous avons la figure 4.4 dans le plan des  $\tau$ - $t$  qui couvre toute la portion supérieure du premier quadrant au-dessus de la droite  $t = \tau$  :

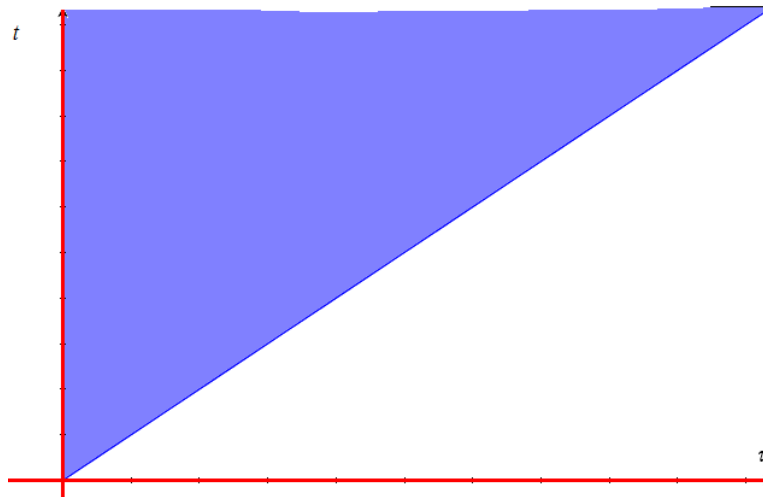


Figure 4.4

Considérant cette région de type II plutôt que de type I, nous pouvons donc permuter l'ordre d'intégration et obtenir en continuant où nous étions rendus

$$\begin{aligned}
 F(s)G(s) &= \int_0^{\infty} g(\tau) \int_{\tau}^{\infty} f(t-\tau)e^{-st} dt d\tau = \int_0^{\infty} g(\tau) \int_0^t f(t-\tau)e^{-st} d\tau dt = \\
 &= \int_0^{\infty} \left( \int_0^t g(\tau)f(t-\tau) d\tau \right) e^{-st} dt = \int_0^{\infty} g(t) * f(t) e^{-st} dt = \mathcal{L}[f(t) * g(t)].
 \end{aligned}$$

**4.9 Exemple fractions partielles/convolution** En réappliquant la propriété b) du théorème 4.6, on peut, par exemple, résoudre une équation différentielle linéaire à coefficients constants en faisant usage de la transformée de Laplace. En effet, considérons le problème

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = x(t), \quad y^{(n)}(0) = y_n$$

où  $y^{(n)}$  désigne la dérivée d'ordre  $n$  de la fonction  $y$  ( $y^{(0)} = y$ ) et où les  $n$  conditions initiales ont été données. Alors si  $Y(s) \leftrightarrow y(t)$ ,  $X(s) \leftrightarrow x(t)$ , on a que l'É.D. est transformée en l'équation algébrique

$$Q(s) + R(s)Y(s) = X(s)$$

où  $Q(s)$  est un polynôme de degré  $n - 1$  en  $s$  qui dépend des conditions initiales et des coefficients de l'équation différentielle et où  $H(s) = 1/R(s)$  est la fonction de transfert. En fait,  $R(s)$  est le polynôme caractéristique de l'É.D. :

$$R(s) = \sum_{k=0}^n a_{n-k} s^k, \quad a_0 = 1.$$

On voit donc que  $Y(s) = \frac{X(s) - Q(s)}{R(s)}$ .

La technique des fractions partielles est fréquemment utilisée pour inverser la dernière fonction. La fonction  $Y$  est toujours rationnelle si l'entrée  $x(t)$  est constituée de combinaisons linéaires et de produits de polynômes, sinus, cosinus, exponentielles. Dans le cas où l'entrée  $x(t)$  est continue par morceaux, il suffit d'appliquer le même principe et la propriété de translation.

En particulier, si les conditions initiales sont nulles (système dit au repos), alors  $Q$  est identiquement nul et la solution de l'É.D. est

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{X(s)}{R(s)}\right] = \mathcal{L}^{-1}[X(s) \cdot H(s)] = x(t) * h(t)$$

où  $h(t) = \mathcal{L}^{-1}[H(s)]$  est dite réponse impulsionnelle. Cette réponse est entièrement caractérisée par les coefficients.

**4.10 Exemple** Développement en fractions partielles. Soit  $\frac{F(s)}{G(s)}$  une fonction rationnelle propre ( $\deg F < \deg G$ ) et où l'on suppose que  $F$  et  $G$  n'ont pas de facteurs communs.

**4.10.1** Si  $a$  est un zéro simple de  $G$ , alors le développement en fraction partielle de  $\frac{F(s)}{G(s)}$  contient un terme de la forme  $\frac{A}{s-a}$  où  $A = \lim_{s \rightarrow a} \frac{(s-a)F(s)}{G(s)}$ .

**4.10.2** Si  $a$  est un zéro d'ordre  $m$  de  $G$ , alors le développement en fraction partielle de  $\frac{F(s)}{G(s)}$  contient des termes de la forme  $\frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \dots + \frac{A_1}{s-a}$  où

$$A_m = \lim_{s \rightarrow a} \frac{(s-a)^m F(s)}{G(s)} \quad \text{et} \quad A_k = \frac{1}{(m-k)!} \lim_{s \rightarrow a} \frac{d^{m-k}}{ds^{m-k}} \left( \frac{(s-a)^m F(s)}{G(s)} \right) \quad (k = 1, 2, \dots, m-1)$$

**4.11 Exemple** Trouvons la transformée de Laplace inverse de la fonction

$$Y(s) = \frac{10(s^2 + 119)}{(s+5)(s^2 + 10s + 169)}$$

Soit  $y(t)$  cette transformée. Sans les nombres complexes, on trouve facilement que

$$Y(s) = \frac{10}{s+5} + \frac{100}{s^2 + 10s + 169}$$

et en complétant le carré on obtient

$$Y(s) = \frac{10}{s+5} - \frac{100}{(s+5)^2 + 12^2} \leftrightarrow 10e^{-5t} - \frac{100}{12} e^{-5t} \sin(12t)$$

Donc,  $y(t) = 10e^{-5t} - \frac{25}{3} e^{-5t} \sin(12t)$ . Avec les nombres complexes, on trouve

$$Y(s) = \frac{10(s^2 + 119)}{(s+5)(s - (-5 + 12i))(s - (-5 - 12i))} = \frac{10}{s+5} + \frac{25i/6}{s - (-5 + 12i)} - \frac{25i/6}{s - (-5 - 12i)}$$

Une table de transformée ainsi que les formules d'Euler permettent d'écrire que

$$y(t) = 10e^{-5t} + \frac{25i}{6}e^{(-5+12i)t} - \frac{25i}{6}e^{(-5-12i)t} = 10e^{-5t} - \frac{25}{3}e^{-5t} \left( \frac{e^{12it} - e^{-12it}}{2i} \right) = 10e^{-5t} - \frac{25}{3}e^{-5t} \sin(12t)$$

**4.12** Trouvons maintenant la transformée de Laplace inverse de la fonction  $Y(s) = \frac{1}{(s^4 + 4)^2}$ .

On a, en factorisant seulement dans les réels,

$$Y(s) = \frac{1}{(s^2 - 2s + 2)^2 (s^2 + 2s + 2)^2} = \frac{1}{((s-1)^2 + 1)^2} \cdot \frac{1}{((s+1)^2 + 1)^2}$$

et donc  $y(t) = (e^t \sin t) * (e^t \sin t) * (e^{-t} \sin t) * (e^{-t} \sin t)$ . Le calcul donne (la convolution est associative)

$$y(t) = \left( \frac{e^t (\sin t - t \cos t)}{2} \right) * \left( \frac{e^{-t} (\sin t - t \cos t)}{2} \right) = \left( \frac{3 \cos t}{128} + \left( \frac{t}{64} + \frac{3}{128} \right) \sin t \right) e^{-t} - \left( \frac{3 \cos t}{128} - \left( \frac{3}{128} - \frac{t}{64} \right) \sin t \right) e^t$$

Un « développement en fractions partielles complexes » peut être fait en informant le système Nspire qu'une variable est considérée complexe et non réelle :

$$\begin{aligned} & \text{cFactor}\left(\frac{1}{(s^4+4)^2}, s\right) \\ & \frac{1}{(s-(1+i))^2 \cdot (s-(1-i))^2 \cdot (s+1-i)^2 \cdot (s+1+i)^2} \\ & \text{expand}\left(\frac{1}{(s-(1+i))^2 \cdot (s-(1-i))^2 \cdot (s+1-i)^2 \cdot (s+1+i)^2}, s\right) \Big|_{s=s_{-}} \\ & \frac{\frac{3}{256} + \frac{3}{256} \cdot i}{s_{-}+1+i} + \frac{\frac{1}{128} \cdot i}{(s_{-}+1+i)^2} + \frac{\frac{3}{256} - \frac{3}{256} \cdot i}{s_{-}+1-i} + \frac{\frac{-1}{128} \cdot i}{(s_{-}+1-i)^2} + \frac{-\left(\frac{3}{256} - \frac{3}{256} \cdot i\right)}{s_{-}-(1-i)} + \frac{\frac{-1}{128} \cdot i}{(s_{-}-(1-i))^2} + \frac{-\left(\frac{3}{256} + \frac{3}{256} \cdot i\right)}{s_{-}-(1+i)} \\ & \textcircled{c} \frac{\frac{3}{256} + \frac{3}{256} * i}{s_{-}+1+i} + \frac{\frac{1}{128} * i}{(s_{-}+1+i)^2} + \frac{\frac{3}{256} - \frac{3}{256} * i}{s_{-}+1-i} + \frac{\frac{-1}{128} * i}{(s_{-}+1-i)^2} + \frac{-\left(\frac{3}{256} - \frac{3}{256} * i\right)}{s_{-}-(1-i)} + \frac{\frac{-1}{128} * i}{(s_{-}-(1-i))^2} + \\ & \frac{-\left(\frac{3}{256} + \frac{3}{256} * i\right)}{s_{-}-(1+i)} + \frac{\frac{1}{128} * i}{(s_{-}-(1+i))^2} \end{aligned}$$

Figure 4.5

**4.13 La convolution et l'importance de la « fonction » de Dirac** L'équation différentielle d'un problème de masse-ressort est

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + k y = f(t)$$

où  $m$  désigne la masse de l'objet,  $b$  la constante d'amortissement,  $k$  la constante de rappel,  $y(t)$  la position de l'objet à l'instant  $t$  et  $f(t)$  la force extérieure. Quant à celle d'un circuit RLC, c'est

$$LC \frac{d^2 v_C}{dt^2} + RC \frac{dv_C}{dt} + v_C = E(t)$$

où  $R, L, C$  sont les composantes branchées en série,  $E(t)$  est la source et on a les relations suivantes :

$$i(t) = C \frac{dv_C}{dt}, \quad v_C = \frac{q}{C}, \quad i = \frac{dq}{dt}.$$

On voit donc qu'il est utile de considérer l'équation différentielle linéaire du deuxième ordre à coefficients constants suivantes et nous allons supposer les conditions initiales nulles (système au repos)

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + c y = x(t), \quad y(0) = y'(0) = 0 \quad (x(t) = 0 \text{ si } t < 0)$$

Souvent, dans la dernière équation différentielle, on dit que  $x(t)$  est l'entrée (« input ») et que  $y(t)$  est la sortie (« output »). En utilisant la transformée de Laplace où  $Y(s) \leftrightarrow y(t)$ ,  $X(s) \leftrightarrow x(t)$ , l'É.D. est transformée en

$$Y(s) = X(s) \cdot H(s)$$

où  $H(s) = \frac{1}{as^2 + bs + c}$  est appelée fonction de transfert (transformée de l'output sur transformée de

l'input). Mais pour obtenir l'égalité  $Y = H$ , il aurait fallu que l'entrée  $x(t)$  soit telle que sa transformée de Laplace  $X$  soit 1. Or, pour une fonction « normale », cela est impossible en vertu du théorème 4.6 a) ... On obtiendra une telle fonction « généralisée » en considérant, par exemple, des limites de fonctions indicatrices d'intervalles. Voyons cela de plus près. Étant donné  $\varepsilon > 0$ , définissons une fonction continue par morceaux et satisfaisant les hypothèses du théorème 4.6 a) comme suit :

$$f_\varepsilon(t) = \begin{cases} 1 & \text{si } 0 < t < \varepsilon \\ \varepsilon & \\ 0 & \text{si } t > \varepsilon \end{cases}$$

On trouve facilement que  $\mathcal{L}[f_\varepsilon(t)] = \frac{1 - e^{-\varepsilon s}}{\varepsilon s}$  et, bien que  $\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(t)$  n'existe pas au sens classique, la limite de la transformée existe et  $\lim_{\varepsilon \rightarrow 0^+} \mathcal{L}[f_\varepsilon(t)] = 1$ . Pour cette raison, la « fonction » de Dirac ou fonction impulsion est introduite est satisfait aux exigences suivantes :

$$\delta(t) = \frac{d}{dt} u(t) = \begin{cases} 0 & \text{si } t \neq 0 \\ \infty & \text{si } t = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1, \quad \int_{-\infty}^{\infty} \delta(t-a)x(t) dt = x(a)$$

où, dans la dernière formule, la fonction  $x$  est supposée continue en  $t = a$ . En d'autres termes, la « fonction » de Dirac  $\delta(t - a)$  concentre la masse en  $t = a$  et cette formule se comprend bien lorsqu'on calcule l'intégrale : en effet,

$$\int_a^{a+\varepsilon} \frac{1}{\varepsilon} (\text{CHI}(t-a) - \text{CHI}(t-a-\varepsilon)) x(t) dt \rightarrow x(a) \text{ si } \varepsilon \rightarrow 0^+.$$

Par conséquent, si  $h(t) \leftrightarrow H(s)$ , alors  $h(t)$  est entièrement caractérisée par les coefficients du problème et il est naturel d'appeler  $h(t)$  la réponse impulsionnelle (la sortie lorsque l'entrée est une impulsion). La solution de l'É.D. est la convolution de l'entrée et de la réponse impulsionnelle :  $y(t) = x(t) * h(t)$ .

C'est à cause de la transformée de Laplace, du fait d'imposer des conditions initiales nulles et de considérer toutes les fonctions nulles avant  $t = 0$  si l'on a pu obtenir le résultat. La linéarité de l'intégrale et la propriété de translation (figure 4.3, deuxième ligne du tableau) font en sorte que le problème est un exemple de ce qu'on appelle un « système linéaire invariant dans le temps ».

**4.14 Exemple** Revenons à l'exemple 3.9.2 où nous cherchions la « fréquence de résonance », i.e. la valeur de  $\omega$  qui procurera une amplitude en régime permanent maximale pour le problème  $y'' + \frac{1}{4}y' + 2y = 2\cos(\omega t)$ . Supposons les c.i. nulles puisque cela n'affecte que le régime transitoire.

Puisque la fonction de transfert est

$$H(s) = \frac{1}{s^2 + \frac{s}{4} + 2} = \frac{1}{\left(s + \frac{1}{8}\right)^2 + \frac{127}{64}},$$

alors la réponse impulsionnelle est

$$h(t) = \frac{8\sqrt{127}}{127} e^{-t/8} \sin\left(\frac{\sqrt{127}}{8}t\right).$$

La convolution de  $h(t)$  avec l'input  $2\cos(\omega t)$  donnera la solution complète dont on pourra extraire la solution particulière et en trouver l'amplitude : une calculatrice symbolique peut très bien finir le travail pour nous :

$h(t) := \frac{8 \cdot \sqrt{127} \cdot e^{-\frac{t}{8}} \cdot \sin\left(\frac{\sqrt{127} \cdot t}{8}\right)}{127}$	Done
$x(t) := 2 \cdot \cos(\omega \cdot t)$	Done
$\int_0^t (x(\tau) \cdot h(t-\tau)) d\tau \rightarrow y(t)$	Done
$\text{propFrac}(\text{tCollect}(y(t)))$	
$\frac{(64 - 32 \cdot \omega^2) \cdot \cos(\omega \cdot t) + \frac{8 \cdot \omega \cdot \sin(\omega \cdot t)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} + \frac{(32 \cdot \omega^2 - 64) \cdot e^{-\frac{t}{8}} \cdot \cos\left(\frac{\sqrt{127} \cdot t}{8}\right) + \left(\frac{-32 \cdot \omega^2 \cdot \sqrt{127}}{127} - \frac{64 \cdot \sqrt{127}}{127}\right) \cdot e^{-\frac{t}{8}}}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}$	
$\Delta \text{fMax}\left(\text{norm}\left(\left[\frac{64 - 32 \cdot \omega^2}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}, \frac{8 \cdot \omega}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}\right], \omega\right) \mid \omega > 0\right)$	$\omega = \frac{3 \cdot \sqrt{14}}{8}$

Figure 4.6

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## Polígonos genéricos – Allgemeine Vielecke – Generic polygons

En geometría, un polígono es una figura geométrica plana y está compuesta por una secuencia finita de segmentos rectos consecutivos que encierran una región en el plano. Estos segmentos son llamados lados, y los puntos en que se intersecan se llaman vértices.

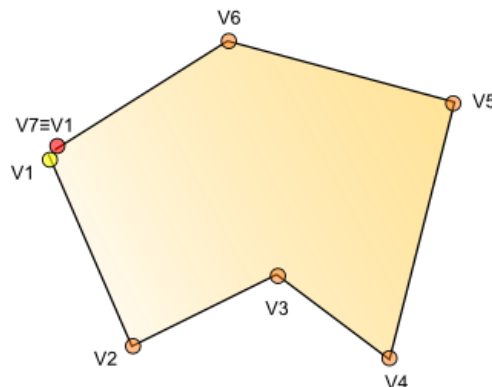
In geometry, a polygon is a plane geometric figure which consists of a finite sequence of consecutive straight segments enclosing a region in the plane. These segments are called sides, and the points at which they intersect are called vertices.

En esta hoja de trabajo:

- un polígono se define por sus vértices;
- la definición de vértices puede ser en sentido horario o antihorario;
- el último vértice debe ser igual al primero;
- los puntos de intersección deben ser numerados dos veces.

This worksheet contains:

- a polygon is defined by its vertices;
- the definition of vertices can be clockwise or counterclockwise;
- the last vertex must coincide the first one;
- intersection points must be numbered twice.



```
#1: LOAD(0:\DOKUS\DNL\DNL126\Polygonos.mth)
```

```
#2: polygon := [ [ 0 156 156 130 130 26 26 78 78 0 0 ],
                 [ 0 0 52 52 26 26 104 104 130 130 0 ] ],
```

## MATRICES AUXILIARES PARA MODIFICAR POLÍGONOS – AUXILIARY MATRICES FOR MODIFYING POLYGONS

### • Matrix Rot

Rotation by angle  $\theta$  around the origin of the Cartesian coordinate system.

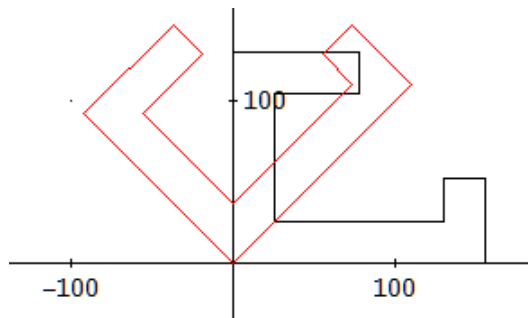
Syntax: Rot( $\theta$ )

$\theta$ : angle of rotation anticlockwise.

$$\#3: \left[ \text{Rot}(\theta), \text{Rot}\left(\frac{\pi}{4}\right) \right] = \left[ \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \right]$$

#4: polygon

#5: polygon · Rot $\left(\frac{\pi}{4}\right)$



### • Matrix Ref

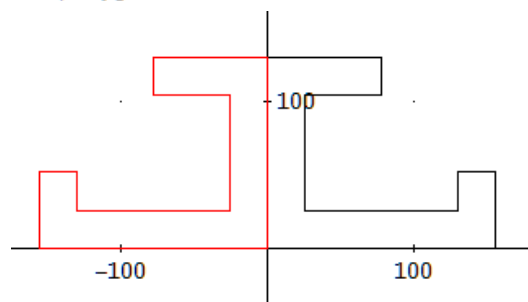
Reflection on a line passing the origin.

Syntax: Ref( $\theta$ )

$\theta$ : angle between the line and the positive x-axis.

$$\#6: \left[ \text{Ref}(\theta), \text{Ref}\left(\frac{\pi}{6}\right) \right] = \left[ \begin{bmatrix} \cos(2 \cdot \theta) & \sin(2 \cdot \theta) \\ \sin(2 \cdot \theta) & -\cos(2 \cdot \theta) \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \right]$$

#7: polygon · Ref( $90^\circ$ )



### • Matrix Sc

Changes the size or scale of a polygon.

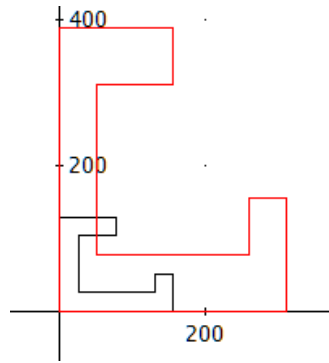
Syntax: Sc( $S_x, S_y$ )

$S_x, S_y$ : scaling factors in x- and y-direction.

$$\#8: [\text{Sc}(S_x, S_y), \text{Sc}(2, 3)] = \left[ \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \right]$$

#9: polygon · Sc(2, 3)





• Matrix Sh

Stretching about the origin of the Cartesian coordinate system (|| to the axes).

Syntax: Sh( $\lambda_x, \lambda_y$ )

$\lambda_x, \lambda_y$ : offset for stretching, parallel to x and parallel to y

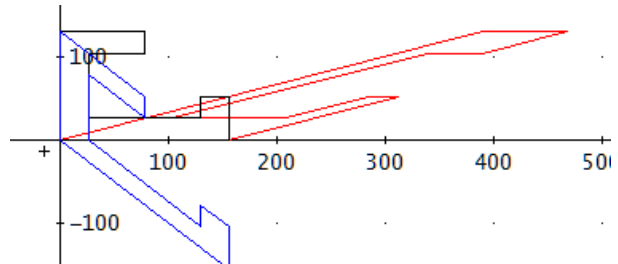
$$\#10: [Sh(\lambda_x, \lambda_y), Sh(3, -2)] = \left[ \begin{bmatrix} 1 & \lambda_y \\ \lambda_x & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \right]$$

This Mapping is not commutative.

#11: polygon

#12: polygon·Sh(3, 0)

#13: polygon·Sh(0, -1)

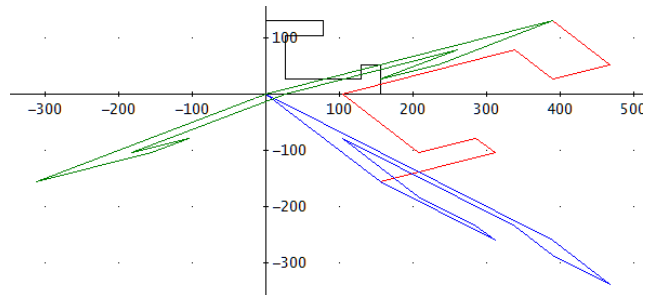


#14: polygon·Sh(3, -1)

#15: polygon·Sh(3, 0)·Sh(0, -1)

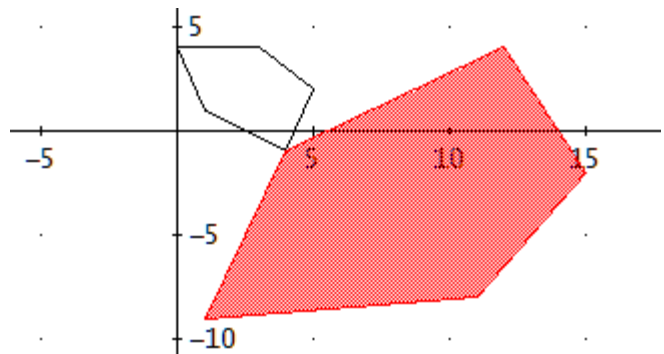
#16: polygon·Sh(0, -1)·Sh(3, 0)

red, blue, green



$$\#14: \text{poly} := \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 5 & 2 \\ 3 & 4 \\ 0 & 4 \\ 1 & 1 \end{bmatrix}$$

#15: POLYGON\_FILL(poly·Sh(3, -2))



## FUNCIONES AUXILIARES PARA MODIFICAR POLÍGONOS – AUXILIARY FUNCTIONS FOR MODIFYING POLIGONS

### • Function Translate

Shifting the polygon by given distances in x- and y-direction.

Syntax: Translate(p,Ux,Uy)

p = matrix of polygon, Ux, Uy = shift in x- and y-direction

```
Translate(P, Ux, Uy, P_) :=
```

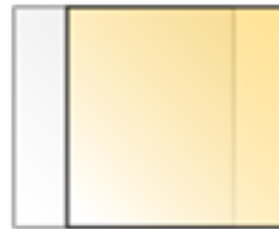
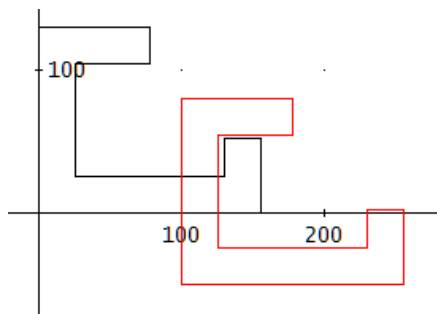
```
  Prog
```

```
#12:   "use coordenadas homogeneas"
```

```
       P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'
```

```
       DELETE((P_·[1, 0, 0; 0, 1, 0; Ux, Uy, 1])', 3)'
```

```
#13: Translate(polygon, 100, -50)
```



### • Function Rotate

Rotate the polygon anticlockwise by the angle  $\theta$ .

Syntax: Rotate(p, $\theta$ )

```
Rotate(P,  $\theta$ , P_) :=
```

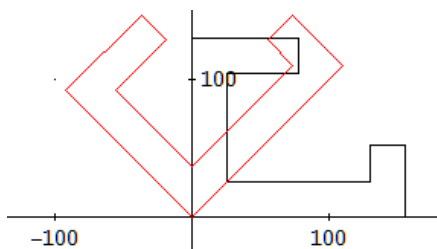
```
  Prog
```

```
#14:   "use coordenadas homogeneas"
```

```
       P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'
```

```
       DELETE((P_·[COS(- $\theta$ ), -SIN(- $\theta$ ), 0; SIN(- $\theta$ ), COS(- $\theta$ ), 0; 0, 0, 1])', 3)'
```

```
#15: Rotate(polygon, 45°)
```



### • Function Scale

Enlarges or reduces the polygon, keeping the same proportions after scaling.

Syntax: Scale(p,Sx,Sy)

```
Scale(P, Sx, Sy, P_) :=
```

```
  Prog
```

```
#16:   "use coordenadas homogeneas"
```

```
       P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'
```

```
       DELETE((P_·[Sx, 0, 0; 0, Sy, 0; 0, 0, 1])', 3)'
```

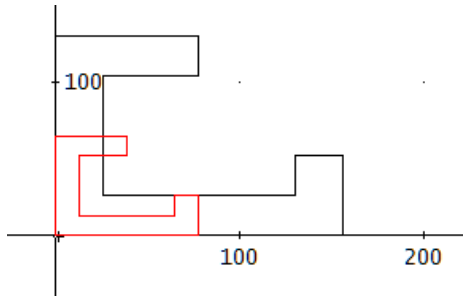
```
#17: Scale(polygon, 0.5, 0.5)
```

### • Function Scale

Enlarges or reduces the polygon, keeping the same proportions after scaling.

Syntax: Scale(p,Sx,Sy)

```
Scale(P, Sx, Sy, P_) :=
  Prog
#16:   "use coordenadas homogeneas"
      P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'
      DELETE((P_·[Sx, 0, 0; 0, Sy, 0; 0, 0, 1])', 3)'
#17: Scale(polygon, 0.5, 0.5)
```



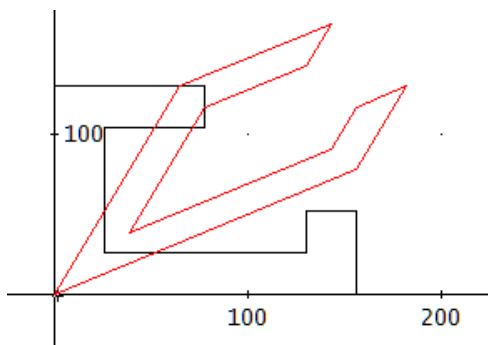
### • Function Shear

Stretches a polygon about the origin of the Cartesian coordinate system.

Syntax: Shear(p,λx,λy)

λx, λy are the stretching factors parallel to x- and y-axis.

```
Shear(P, λx, λy, P_) :=
  Prog
#18:   "use coordenadas homogeneas"
      P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'
      DELETE((P_·[1, λy, 0; λx, 1, 0; 0, 0, 1])', 3)'
#19: Shear(polygon, 0.5, 0)
```



### • Function Mirror

Creates the symmetry of a polygon.

Syntax: Mirror(p,θ)

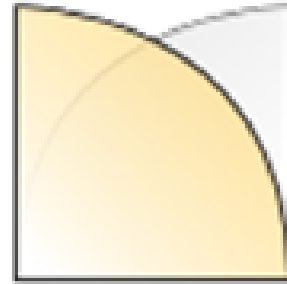
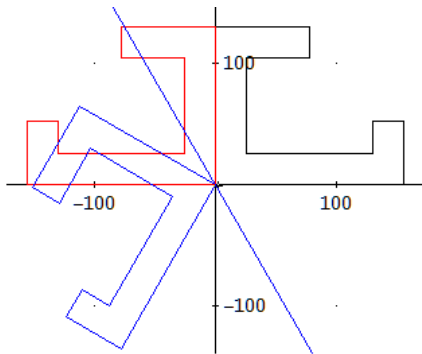
θ: angle of the line of reflection through the origin with the name of the x-axis or the name of the case-insensitive axes ("x", "X", "y", "Y", "xy", "XY").

```
Mirror(P, θ, P_, α, θa, k) :=
  Prog
  "use coordenadas homogeneas"
  If STRING?(θ)
  Prog
#27:   θa := ["x", "X", "y", "Y", "xy", "XY"; 0, 0, π/2, π/2, π/4, π/4]
      VECTOR(IF(θ = ELEMENT(θa, 1, k), α := ELEMENT(θa, 2, k), 0), k, 1, DIM(θa'))
      α := θ
      P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'
      DELETE((P_·[COS(2·α), SIN(2·α), 0; SIN(2·α), -COS(2·α), 0; 0, 0, 1])', 3)'
```

#28: `Mirror(polygon, y)`

#29: `Mirror(polygon,  $\frac{2 \cdot \pi}{3}$ )`

#30: `x * TAN( $\frac{2 \cdot \pi}{3}$ )`



### • Función Trim

Cuts the symmetry of a polygon.

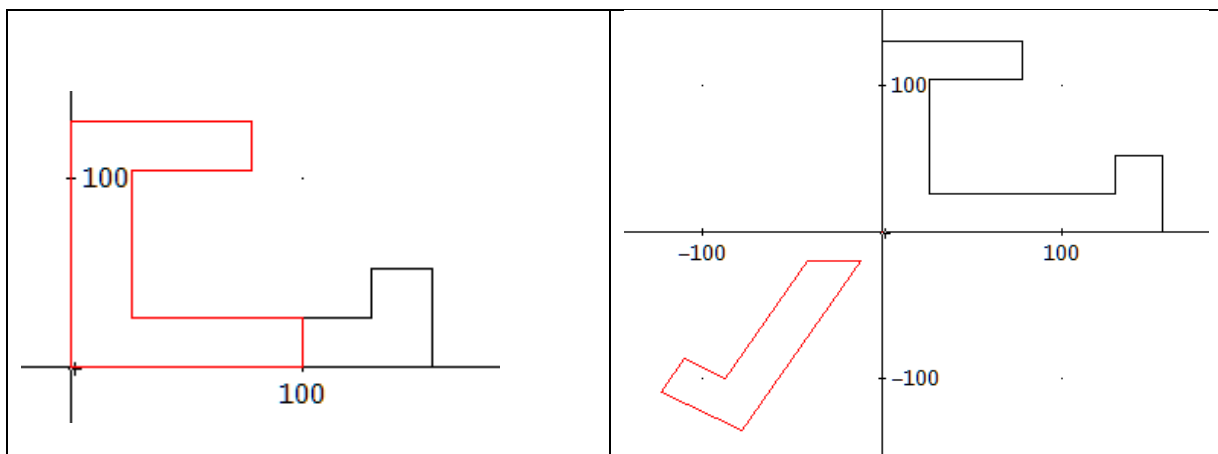
Syntax: `Trim(p, TLV, zone)`

TLV: threshold limit value,

zone: half plane (relative) to maintain ["y+", "y-", "x+", "x-"].

#24: `Trim(polygon, 100, x-)`

#25: `Trim(Mirror(polygon,  $\frac{2 \cdot \pi}{3}$ ), -20, y-)`



```

Trim(pol, TLV, zone, TLV_, pgon_, chunk_, z, Δθ, r, j, k) :=
PROG(
  z := ["y+", 0, 1; "y-", π, -1; "x+", π/2, 1; "x-", - π/2, -1],
  k := 1,
  LOOP(
    IF(z↓k↓1 = zone,
      PROG(
        [Δθ := z↓k↓2, TLV_ := TLV·z↓k↓3, pgon_ := pol·Rot(Δθ)],
        exit),
      0),
    IF(k = 4, exit, k := + 1)),
  chunk_ := [],
  [r := DIM(pol), j := 1, k := 1],
  LOOP(
    IF(pgon_↓k↓2 ≥ TLV_,
      PROG(
        chunk_ := INSERT([pgon_↓k↓1, pgon_↓k↓2], chunk_, j),
        j := + 1),
      0),
    IF((pgon_↓k↓2 - TLV_)·(pgon_↓(k + 1)↓2 - TLV_) < 0,
      PROG(
        chunk_ := INSERT([(pgon_↓(k + 1)↓1 - pgon_↓k↓1)·(TLV_ - pgon_↓k↓2)
          / (pgon_↓(k + 1)↓2 - pgon_↓k↓2) + pgon_↓k↓1, TLV_], chunk_, j),
        j := + 1), 0),
    IF(k = r - 1, exit, k := + 1)),
  chunk_ := chunk_·Rot(-Δθ),
  chunk_ := INSERT([chunk_↓1↓1, chunk_↓1↓2], chunk_, j),
  chunk_)

```

## FUNCIONES PARA GENERAR POLIGONOS PLANOS - FUNCTIONS TO CREATE PLANE POLYGONS

### • Function pgon

Generates regular convex polygons.

Syntax: pgon(s,c,θ)

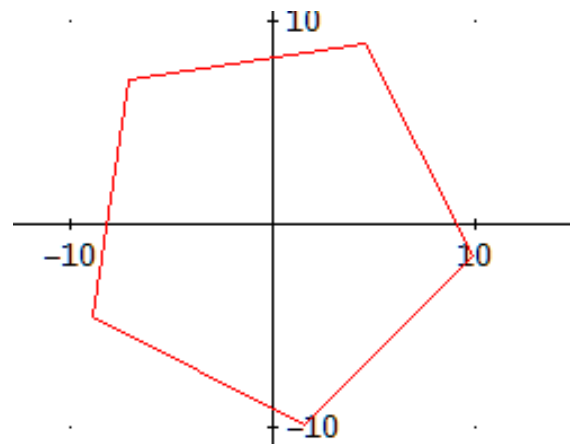
s: number of vertices, c: radius, θ: angle of rotation anticlockwise

```

pgon(s, c, θ, out_) :=
  Prog
  s := MAX(s, 3)
#27: out_ := VECTOR([c·SIN(2·π·k/s), c·COS(2·π·k/s)], k, 0, s)
  out_ := out_·Rot(θ)
  out_

```

#28:  $\text{pgon}\left(5, 10, \frac{\pi}{4}\right)$



### • Función hPgon

Generates hollow convex regular polygons.

Syntax: hPgon(s,c,g,θ)

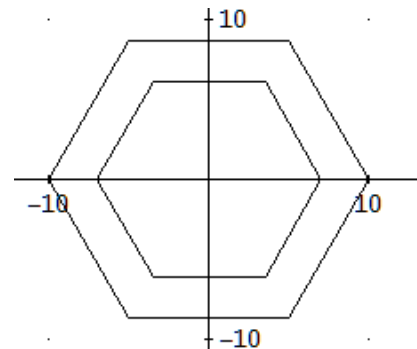
s: number of vertices, c: outer radius, g: thickness of border,  
θ: angle of rotation anticlockwise

```

hPgon(s, c, g, θ, out_) :=
  Prog
  out_ := APPEND(pgon(s, c, 0), REVERSE(pgon(s, c - g, 0)))
#29: out_ := APPEND(out_, [FIRST(out_)])
  out_ := out_.Rot(θ)
  out_

#30: hPgon(6, 10, 3, 90°)

```



### • Function oPgon

Generates superposition of regular convex polygons.

Syntax: oPgon(s,ce,ci,θ)

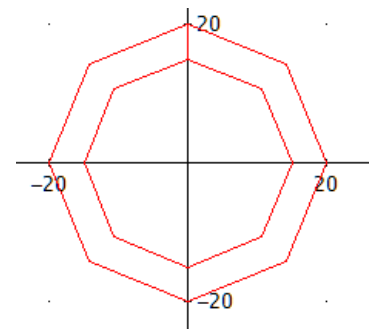
s: number of vertices, ce: outer radius, ci: inner radius,  
θ: angle of rotation anticlockwise

```

oPgon(s, ce, ci, θ, out_) :=
  Prog
  out_ := APPEND(pgon(s, ce, 0), pgon(s, ci, 0))
#31: out_ := APPEND(out_, [FIRST(out_)])
  out_ := out_.Rot(θ)
  out_

#32: oPgon(8, 20, 15, 0)

```



### • Function sPgon

Generates regular and complex polygons.

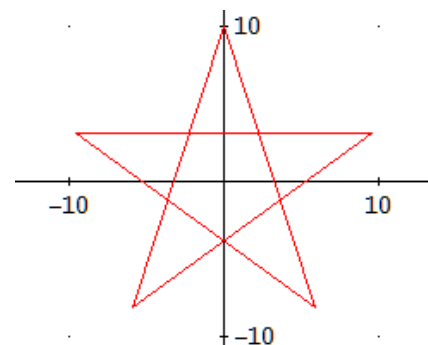
Syntax: sPgon(p,q,c,θ)

p: number of vertices or sides ( $\geq 3$ ),  
q: spacing of connected vertices ( $\{p / q\}$  is the Schläfli polygon symbol),  
c: radius, θ: angle of rotation anticlockwise

The Schläfli symbol is named after the 19th-century Swiss mathematician Ludwig Schläfli (see more in Wikipedia).

*Regular or starry polygon stellations are for drawing purposes only.*

```
#34: sPgon(5, 5/2, 10, 0)
```

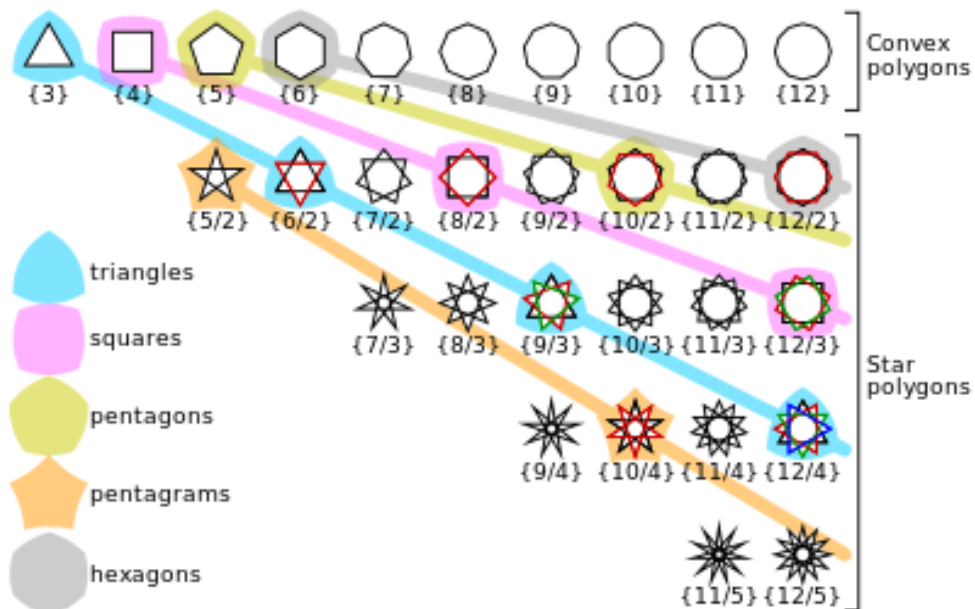


```

sPgon(p, q, c,  $\theta$ , q_, k := 0, out_ := [], out, GCD_, star_) :=
  Prog
  q_ := MIN(q, CEILING(p/2) - 1)
  If p  $\geq$  3
  Prog
  If GCD_AUX(p, q_) = 1  $\wedge$  q_ > 1
  Prog
  Loop
  out_ := INSERT([c·SIN(2· $\pi$ ·k·q_/p), c·COS(2· $\pi$ ·k·q_/p)], out_, k + 1)
  If k = p
  exit
  k := + 1
  out_ := out_·Rot( $\theta$ )
  out_ := APPEND(out_, [FIRST(out_)])
  RETURN out_
  Prog
  "Estrellamiento"
  GCD_ := GCD_AUX(p, q_)
  If GCD_/q_ = 1
  Prog
  "Estrellamiento de un poligono regular"
  Loop
  k := + 1
  out_ := APPEND(out_, [pgon(CEILING(p/q_), c, 2· $\pi$ /p·(k - 1))])
  If k = q_ exit
  RETURN out_
  Prog
  "Estrellamiento de un poligono estrellado"
  If q_/GCD_ - FLOOR(q_/GCD_) = 0
  Prog
  star_ := sPgon(p/GCD_, GCD_, c,  $\theta$ )
  Loop
  k := + 1
  out_ := APPEND(out_, [star_·Rot(2· $\pi$ /p·(k - 1))])
  If k = GCD_ exit
  RETURN out_
  0
  0

```

0



### • Function **esPgon**

Generates the contour of a complex polygon.

Syntax: `esPgon(p,q,c,θ)`

p: number of vertices or sides ( $\geq 3$ ),

q: spacing of connected vertices ( $\{p / q\}$  is the Schläfli polygon symbol),

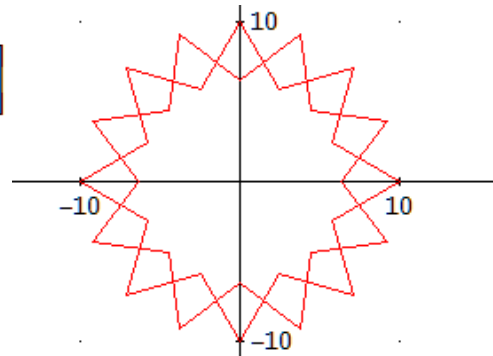
c: radius,  $\theta$ : angle of rotation anticlockwise

```

esPgon(p, q, c, θ, k := 0, ci, r) :=
  Prog
  q_ := MIN(q, CEILING(p/2) - 1)
  Prog
  If p ≥ 3
  Prog
  out_ := []
  ci := c·SIN((p/q_ - 2)·π·q_/(2·p))/SIN(π - (p/q_ - 2)·π·q_/(2·p) - 2·π/(2·p))
  Loop
#35:   r := IF(k/2 - FLOOR(k/2) = 0, c, ci)
        out_ := INSERT([r·SIN(2·π·k/(2·p)), r·COS(2·π·k/(2·p))], out_, k + 1)
        If k = 2·p
          exit
          k := + 1
        out_ := out_.Rot(θ)
        out_ := APPEND(out_, [FIRST(out_)])
  RETURN out_
0

```

#36:  $\left[ \text{esPgon}\left(8, \frac{8}{3}, 10, 0\right), \text{esPgon}\left(8, \frac{8}{3}, 10, \frac{\pi}{8}\right) \right]$



### • Function **ssPgon**

Generates star polygons.

Syntax: `ssPgon(s,cθ,ci,θ)`

s: number of peaks, co: outer radius, ci: inner radius,

$\theta$ : angle of rotation anticlockwise

```

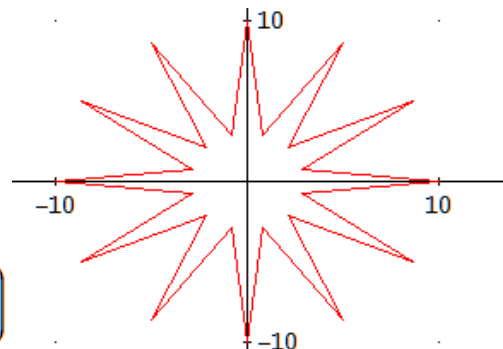
ssPgon(s, co, ci, θ, r, k := 0) :=
  Prog
  If s ≥ 3
  Prog
  out_ := []
  Loop
#37:   r := IF(k/2 - CEILING(k/2) = 0, co, ci)
        out_ := INSERT([r·SIN(2·π·k/(2·s)), r·COS(2·π·k/(2·s))], out_, k + 1)
        If k = 2·s
          exit
          k := + 1
        out_ := out_.Rot(θ)
        out_ := APPEND(out_, [FIRST(out_)])
  RETURN out_
0

```

#38: `ssPgon(12, 10, 3, 0)`

#39: `ssPgon(5, 10, 5, 0)`

#40:  $\text{VECTOR}\left(\text{ssPgon}\left(5, 10, 5, k\right), k, 0, 2\cdot\pi, \frac{\pi}{3}\right)$



(see front cover)



## Cello Tangents of Quartic – and other – Polynomials

Robert Haas and Josef Böhm

In his leisure times Robert likes to play cello. Seeing the bow resting on top of the cello (photo) he wanted to "use easy calculus and linear algebra to analyze such *Cello Tangents* in the simplest case that the curve is a quartic polynomial." He published his investigation in the *Journal of Humanistic Mathematics*<sup>[1]</sup>. This is not only a nice task for students but also inspiration for extending the problem.



Philip Choi's photo

I start reproducing Robert's example:

The half profile of the cello is given by a quartic which can be scaled with a leading coefficient -1. Then we assume two points A(-1/0.4) and B(1/0.8) where the cello bow is resting on the cello. What is the equation of the profile of the instrument?

Since the connecting line of A and B is the common tangent, its slope must be the first derivative of the requested function  $f(x)$  in both points which is  $f'(-1) = f'(1) = 0.2$ .

$$\#1: f(x) := -x^4 + a \cdot x^3 + b \cdot x^2 + c \cdot x + d$$

$$\#2: f'(x) := -4 \cdot x^3 + 3 \cdot a \cdot x^2 + 2 \cdot b \cdot x + c$$

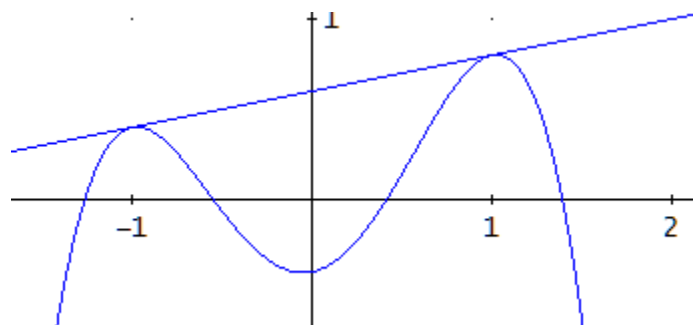
$$\#3: (\text{SOLUTIONS}([f(-1) = 0.4, f(1) = 0.8, f'(-1) = 0.2, f'(1) = 0.2], [a, b, c, d])) \quad 1$$

$$\#4: \left[ 0, 2, \frac{1}{5}, -\frac{2}{5} \right]$$

$$\#5: -x^4 + 2 \cdot x^2 + \frac{1}{5} \cdot x - \frac{2}{5}$$

$$\#6: y - 0.4 = \frac{0.8 - 0.4}{2} \cdot (x + 1)$$

$$\#7: y - \frac{2}{5} = \frac{x + 1}{5}$$



[1] Robert Haas Ph.D., "Cello Tangents of Quartic Polynomials," *Journal of Humanistic Mathematics*, Volume 8 Issue 1 (January 2018), pages 329-331. DOI: 10.5642/jhummath.201801.15. Available at: <https://scholarship.claremont.edu/jhm/vol8/iss1/15>

Robert offers a more general solution involving two parameters: the given points are  $f(-1) = r$  and  $f(1) = s$ . Then the curve results as  $f(x) = -x^4 + 2x^2 + \frac{s-r}{2}x + \frac{s+r-2}{2}$ .

If you are a regular reader of the DNL you will know my preference for applying sliders. So, it is clear that I liked to introduce more parameters to be realized by sliders:

$$\#8: \text{sys} := \left[ f(u) = r, f(v) = s, f_{-}(u) = \frac{s-r}{v-u}, f_{-}(v) = \frac{s-r}{v-u} \right]$$

$$\#9: (\text{SOLUTIONS}(\text{sys}, [a, b, c, d]))_1$$

$$\#10: \left[ 2 \cdot (u+v), -u^2 - 4 \cdot u \cdot v - v^2, \frac{r-s + 2 \cdot u \cdot v \cdot (u+v) \cdot (u-v)}{u-v}, \frac{r \cdot v - u \cdot (s - u \cdot v \cdot (u-v))}{v-u} \right]$$

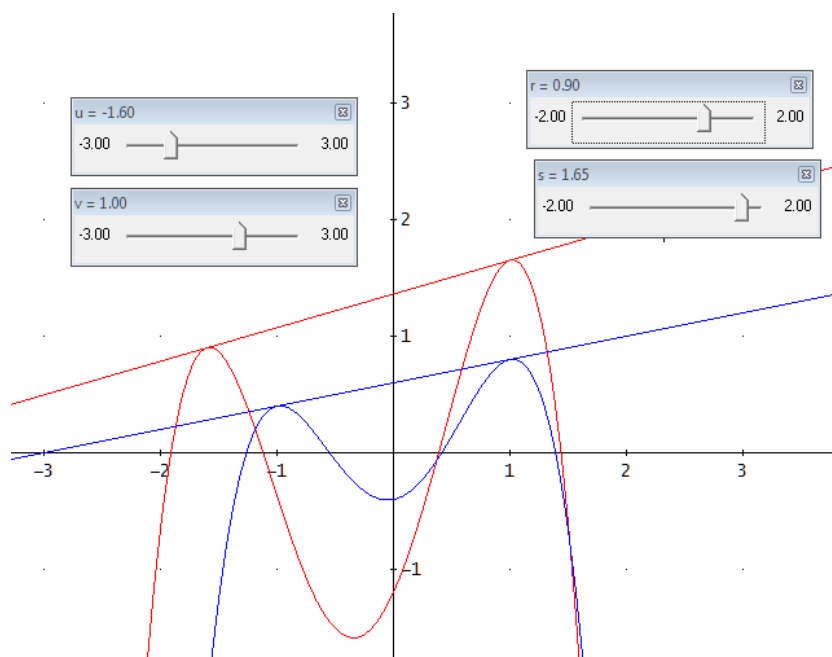
$$\#11: ff(x) := -x^4 + 2 \cdot (u+v) \cdot x^3 + (-u^2 - 4 \cdot u \cdot v - v^2) \cdot x^2 + \frac{r-s + 2 \cdot u \cdot v \cdot (u+v) \cdot (u-v)}{u-v} \cdot x + \frac{r \cdot v - u \cdot (s - u \cdot v \cdot (u-v))}{v-u}$$

$$\#12: y = \frac{s-r}{v-u} \cdot (x-u) + r$$

$$\#13: -x^4 + 2 \cdot x^2 + 0.2 \cdot x - 0.4$$

$$\#14: y = \frac{x+3}{5}$$

The plot below shows the original Robert's "Cello + Bow" and one other Figure generated by four slider bars for  $u$ ,  $v$ ,  $r$  and  $s$ :



That could have been the full story – if I had not the idea to reverse the task. Robert reconstructed the profile of the cello using two tangent points. Let's do it the other way around: Provide the profile curve and find the “double tangent”.

$$\#1: \quad y_-(x) := -x^4 + 3x^2 + x$$



slopes of the tangents for  $x = a$  and  $x = b$  must be equal

$$\#2: \quad y_-'(a) - y_-'(b) = 0$$

$$\#3: \quad -4a^3 + 6a + 4b^3 - 6b = 0$$

$$\#4: \quad 2(a - b) \cdot (2a^2 + 2ab + 2b^2 - 3) = 0$$

Tangent at  $x = a$

$$\#5: \quad y - y_-(a) = y_-'(a) \cdot (x - a)$$

must pass point on curve  $(b, y_-(b))$

$$\#6: \quad y_-(b) - y_-(a) = y_-'(a) \cdot (b - a)$$

$$\#7: \quad y_-(b) - y_-(a) - y_-'(a) \cdot (b - a) = 0$$

$$\#8: \quad -3a^4 + 4a^3 \cdot b + 3a^2 \cdot b^2 - 6a \cdot b^3 - b^4 + 3b^2 = 0$$

$$\#9: \quad (a - b)^2 \cdot (3a^2 + 2ab + b^2 - 3) = 0$$

$(a - b)$  can be removed because  $a = b$  is a trivial solution (the inflection points)

$$\#10: \quad \text{SOLUTIONS}(2a^2 + 2ab + 2b^2 - 3 \wedge 3a^2 + 2ab + b^2 - 3, [a, b])$$

$$\#11: \quad \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} \\ -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} \end{bmatrix}$$

Again, I can remove the solutions with  $a = b$ , then two solutions are remaining.

The Inflection points:

$$\#12: \begin{bmatrix} \frac{\sqrt{2}}{2} & y_{-\left(\frac{\sqrt{2}}{2}\right)} \\ -\frac{\sqrt{2}}{2} & y_{-\left(-\frac{\sqrt{2}}{2}\right)} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} + \frac{5}{4} \\ -\frac{\sqrt{2}}{2} & \frac{5}{4} - \frac{\sqrt{2}}{2} \end{bmatrix}$$

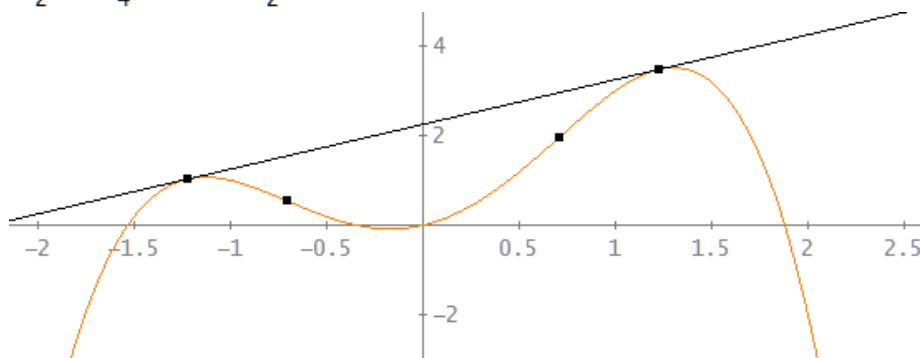
And these are the points we are looking for:

$$\#13: \begin{bmatrix} \frac{\sqrt{6}}{2} & y_{-\left(\frac{\sqrt{6}}{2}\right)} \\ -\frac{\sqrt{6}}{2} & y_{-\left(-\frac{\sqrt{6}}{2}\right)} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} + \frac{9}{4} \\ -\frac{\sqrt{6}}{2} & \frac{9}{4} - \frac{\sqrt{6}}{2} \end{bmatrix}$$

the connecting line gives the common tangent

$$\#14: y - y_{-\left(\frac{\sqrt{6}}{2}\right)} = \frac{y_{-\left(-\frac{\sqrt{6}}{2}\right)} - y_{-\left(\frac{\sqrt{6}}{2}\right)}}{-\frac{\sqrt{6}}{2} - \frac{\sqrt{6}}{2}} \cdot \left(x - \frac{\sqrt{6}}{2}\right)$$

$$\#15: y - \frac{\sqrt{6}}{2} - \frac{9}{4} = x - \frac{\sqrt{6}}{2}$$



I try the same procedure applied on a quintic (and expect more complicated equations, of course):

$$\#16: z(x) := 0.01 \cdot x^5 - 0.045 \cdot x^4 - 0.19 \cdot x^3 + 0.6 \cdot x^2$$

$$\#17: z'(a) - z'(b) = 0$$

$$\#18: \frac{5 \cdot a^4 - 18 \cdot a^3 - 57 \cdot a^2 + 120 \cdot a - b \cdot (5 \cdot b^3 - 18 \cdot b^2 - 57 \cdot b + 120)}{100} = 0$$

$$\#19: (a - b) \cdot (5 \cdot a^3 + a^2 \cdot (5 \cdot b - 18) + a \cdot (5 \cdot b^2 - 18 \cdot b - 57) + 5 \cdot b^3 - 18 \cdot b^2 - 57 \cdot b + 120) = 0$$

$$\#20: eq1 := 5 \cdot a^3 + a^2 \cdot (5 \cdot b - 18) + a \cdot (5 \cdot b^2 - 18 \cdot b - 57) + 5 \cdot b^3 - 18 \cdot b^2 - 57 \cdot b + 120$$

#21:  $\text{FACTOR}(z(b) - z(a) - z'(a) \cdot (b - a) = 0)$

#22:  $(a - b)^2 \cdot (8 \cdot a^3 + a^2 \cdot (6 \cdot b - 27) + a \cdot (4 \cdot b^2 - 18 \cdot b - 76) + 2 \cdot b^3 - 9 \cdot b^2 - 38 \cdot b + 120) = 0$

#23:  $(a - b)^2 \cdot (8 \cdot a^3 + a^2 \cdot (6 \cdot b - 27) + a \cdot (4 \cdot b^2 - 18 \cdot b - 76) + 2 \cdot b^3 - 9 \cdot b^2 - 38 \cdot b + 120) = 0$

#24:  $\text{eq2} := 8 \cdot a^3 + a^2 \cdot (6 \cdot b - 27) + a \cdot (4 \cdot b^2 - 18 \cdot b - 76) + 2 \cdot b^3 - 9 \cdot b^2 - 38 \cdot b + 120$

#25:  $\text{SOLUTIONS}(\text{eq1} = 0 \wedge \text{eq2} = 0, [a, b])$

This does not work - let's take the Groebner Basis tool.

#26:  $\text{GROEBNER\_BASIS}([\text{eq1}, \text{eq2}], [a, b])$

#27: 
$$\left[ \begin{aligned} &8000 \cdot b^9 - 64800 \cdot b^8 - 70240 \cdot b^7 + 1384656 \cdot b^6 - 361876 \cdot b^5 - 10155246 \cdot b^4 + 3332246 \cdot b^3 + \\ &26160285 \cdot b^2 - 3497775 \cdot b - 12421500, 594026480275 \cdot a + 637392000 \cdot b^8 - 20511231200 \cdot b^7 + \\ &92309385840 \cdot b^6 + 253454794424 \cdot b^5 - 1482306777636 \cdot b^4 - 955425073264 \cdot b^3 + 5990839666896 \cdot b^2 \\ &+ 1533887353265 \cdot b - 4750329159600 \end{aligned} \right]$$

#28:  $\text{SOLUTIONS}(8000 \cdot b^9 - 64800 \cdot b^8 - 70240 \cdot b^7 + 1384656 \cdot b^6 - 361876 \cdot b^5 - 10155246 \cdot b^4 + 3332246 \cdot b^3 + 26160285 \cdot b^2 - 3497775 \cdot b - 12421500 = 0, b)$

#29:  $\text{NSOLUTIONS}(8000 \cdot b^9 - 64800 \cdot b^8 - 70240 \cdot b^7 + 1384656 \cdot b^6 - 361876 \cdot b^5 - 10155246 \cdot b^4 + 3332246 \cdot b^3 + 26160285 \cdot b^2 - 3497775 \cdot b - 12421500 = 0, b)$

The first component of expression #27 is an equation of order 9 for variable  $b$ . NSOLUTIONS presents nine distinct real solutions.

#30:  $[3.786812234, 4.682341741, 4.733066917, -1.914440887, -2.798857659, -2.66343206, -0.722164261, 0.8276286527, 2.169045321]$

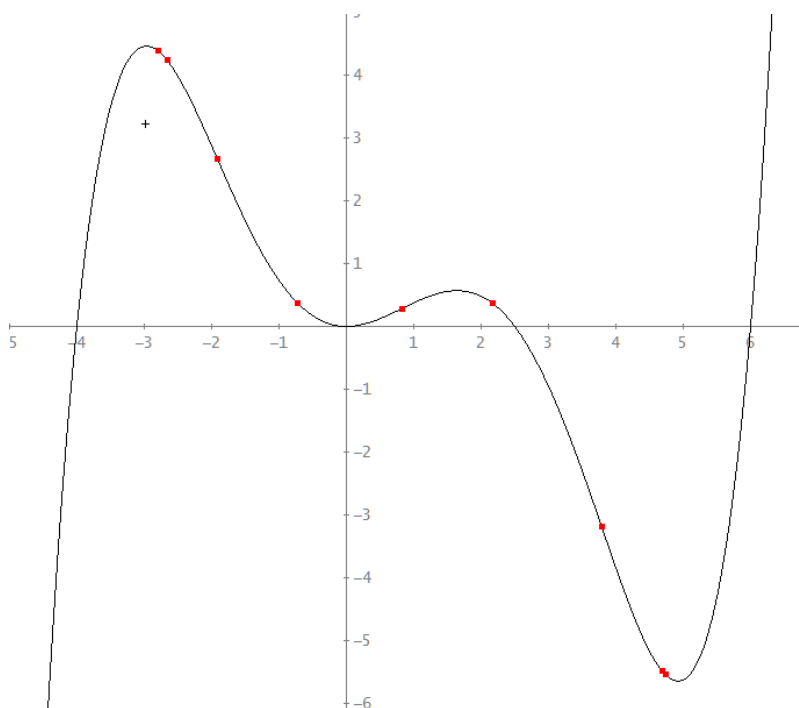
#31:  $\text{sol}s\_b := [3.786812234, 4.682341741, 4.733066917, -1.914440887, -2.798857659, -2.66343206, -0.722164261, 0.8276286527, 2.169045321]$

#32:  $\text{VECTOR}([x, z(x)], x, \text{sol}s\_b)$

this command gives all candidates

Line #32 gives a list of nine points on the graph of  $f(x)$ . They are all candidates for osculating points of double tangents.

Next page shows the graph together with these nine points in red color.



Oops, the inflection points are among them. I should have expected this, because the tangents in the inflection points are among the requested tangents. I use a function for creating the tangent in any point on the curve – and #34 gives the tangent in one point of inflection.

```
#33: tang(b) := y - z(b) = z'(b)·(x - b)
```

```
#34: tang(3.786812234)
```

Yes, this is one tangent in a point of inflection (in red);

I look for all inflection points and remove them from my list of candidates.

```
#35: NSOLUTIONS( ((d/dx)^2 z(x) = 0, x)
```

```
#36: [3.786812234, -1.914440887, 0.8276286527]
```

```
#37: sols_b := [4.682341741, 4.733066917, -2.798857659, -2.66343206, -0.722164261, 2.169045321]
```

```
#38: tang(sols_b )
      1
```

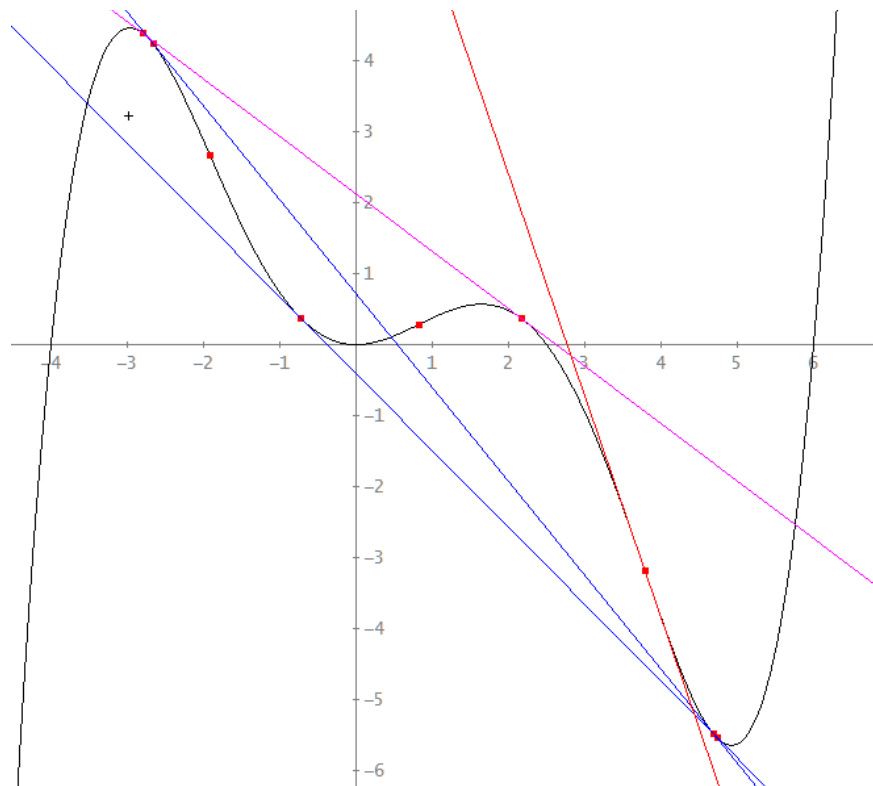
```
#39: tang(sols_b )
      2
```

```
#40: tang(sols_b )
      3
```

```
#41: tang(sols_b )
      4
```

First one (blue), second one (again blue), third one (magenta) – then the tangents repeat.

So, finally we have found three double tangents. See the plot next page.



The inflection points could have been removed earlier by equating  $a$  and  $b$  in eq1 or eq2 and solving the respective equation.

#42: SUBST(eq1, a, b)

#43:  $20 \cdot b^3 - 54 \cdot b^2 - 114 \cdot b + 120$

#44: NSOLUTIONS( $20 \cdot b^3 - 54 \cdot b^2 - 114 \cdot b + 120$ , b)

#45: [3.786812234, -1.914440887, 0.8276286527]

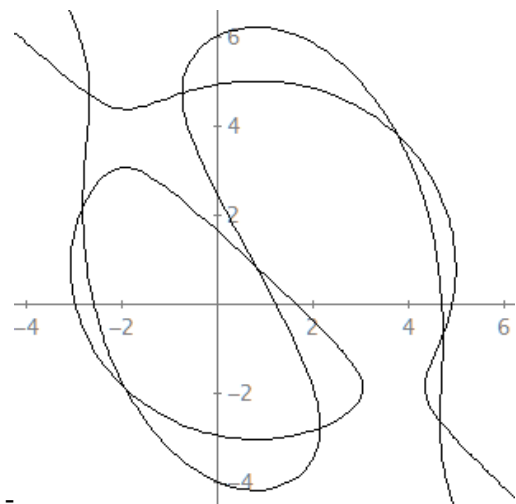
Plotting both implicit functions  $eq1 = 0$  and  $eq2 = 0$  with variables  $a$  and  $b$  results in graph with nine intersection points

#51: eq1 = 0

#52: eq2 = 0

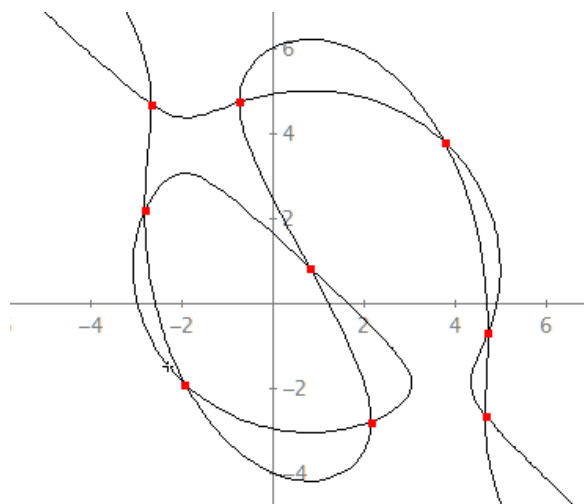
I don't print the calculation which is not difficult, but takes a lot of space. Just substitute the values for  $b$  in the second component of #27 and solve for  $a$ .

Every point represents one pair of osculating points giving one double tangent.



#60:

3.786812245	3.786812234
-2.663432081	4.68234174
-0.7221642883	4.733066917
-1.914440891	-1.914440886
2.169045324	-2.798857659
4.682341749	-2.663432059
4.733066918	-0.722164261
0.8276286535	0.8276286526
-2.798857666	2.16904532



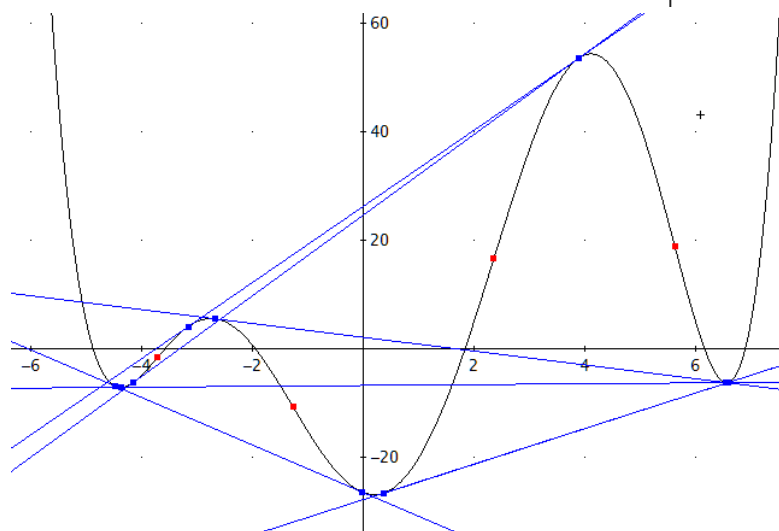
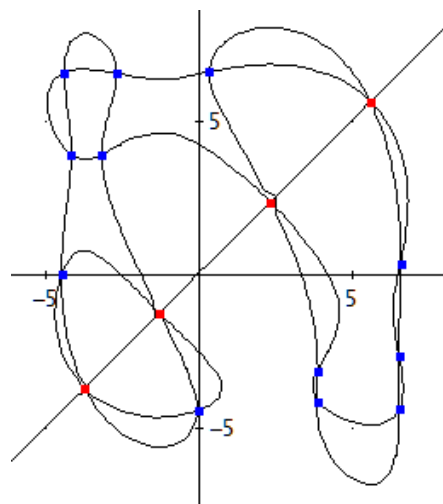
The points of inflection are lying on line  $y = x$ .

Now I became cocky and applied the same procedure with a 6<sup>th</sup> order polynomial:

#1:  $zz(x) := 0.005 \cdot (2 \cdot x^6 - 9 \cdot x^5 - 109 \cdot x^4 + 285 \cdot x^3 + 1871 \cdot x^2 - 780 \cdot x - 5300)$

Since the expressions are very bulky, I refrain from showing the calculations. You can follow them in the file Cello\_6.dfw.

You might observe that there are more intersection points than tangents. Some points represent the same tangent because  $a$  and  $b$  can be interchanged.





## „Parallele“ Ellipsen – “Parallel” Ellipses

Wolfgang Alvermann, Germany

*Heinrich Hemme*

*Heureka – Mathematische Rätsel 2021 – Tageskalender, Anaconda Verlag*

Der 26. Januar eines Kalenders gab Anlass zu vielfältigen Berechnungen und weckte Erinnerungen an den Studienbeginn vor 51 Jahren, als eigentlich alles händisch berechnet werden musste und als Hilfsmittel lediglich ein Rechenschieber und Tabellenwerke verfügbar waren.

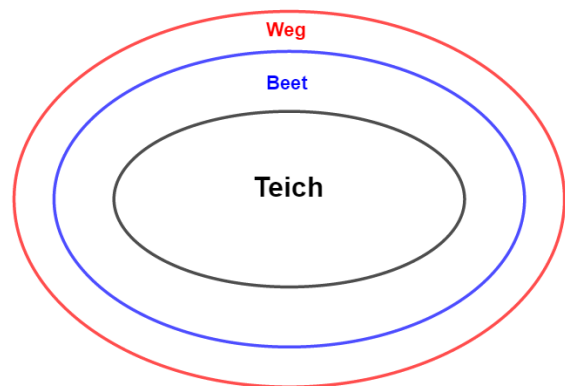
The 26<sup>th</sup> of January of a calendar inspired to a variety of calculations and brought back memories of the beginning of study 51 years ago, when actually everything had to be calculated by hand and only a slide rule and log/trig tables were available as auxiliary tools.

### Logarithmentabelle

#### Der 26. Januar

„In einem Park befindet sich ein Teich mit einer Oberfläche von 121 m<sup>2</sup>. Er hat die Form einer Ellipse, deren Haupt- und Nebenachse im Verhältnis 2:1 stehen.

Um den Teich herum ist ein drei Meter breites Blumenbeet angelegt worden, und um das Blumenbeet und den Teich führt ein Weg, der zwei Meter breit ist. Wie groß ist der Flächeninhalt der größten Ellipse?“



#### 26 January

"In a park there is a pond with a surface of 121 m<sup>2</sup>. It has the shape of an ellipse, whose major and minor axis are in the ratio 2:1. Around the pond a three-meter wide flower bed has been created, and around the flower bed and the pond there is a path that is two meters wide. What is the area of the largest ellipse?"

Die Lösung dieser Frage soll hier zunächst keine Rolle spielen, dafür rücken andere Aufgabenstellungen in den Vordergrund wie z. B.

- In Tabellenwerken ist die Fläche einer Ellipse der Form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  angegeben mit  $A = ab\pi$ .  
Zeigen Sie dies mit Mitteln der Integralrechnung.
- Bestimmen Sie die Größe der Haupt- und Nebenachse gemäß den oben gemachten Angaben.
- Verlängert man die Haupt- und Nebenachsen um 3 bzw. 5 m, erhält man ein Bild wie oben dargestellt. Hat die blaue Ellipse überall den Abstand 3m von dem Teichufer und ist der Weg überall 2 m von dem Beet entfernt? Beantworten Sie diese Fragen durch eine aussagekräftige GeoGebra- bzw. Nspire-Zeichnung! Beschreiben Sie dazu Ihr Vorgehen.

#### Anmerkungen

Inhalte wie Partialbruchzerlegung, Integration mittels Substitution, partielle Integration sind in heutigen Schulbüchern nicht mehr zu finden. Auch lassen sich viele Kurven besser in Parameterform darstellen. Die Möglichkeit, auch darauf einzugehen, war vor Jahren durch Beschränkung auf

zwei von drei Lerngebieten (Analysis, Lineare Algebra und Stochastik) und einer gleichzeitigen höheren Wochenstundenzahl in Leistungskursen gegeben, zumal diese Darstellungsform die Möglichkeit bietet, die bis dahin behandelten Funktionsklassen zu wiederholen und zu festigen.

The solution of this question shall not play a role here for the time being, but other tasks come to the fore, e.g.

- In textbooks the area of an ellipse of form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is given as  $A = ab\pi$ . Show this by means of integration.
- Find the length of the major and minor axes according to the information given above.
- If one extends the major and minor axes by 3 and 5 m, respectively, one gets a picture like the one given above. Is the distance of the blue ellipse from the pond bank 3m and is the path 2m wide everywhere? Give answers by developing a meaningful GeoGebra or Nspire drawing! Describe your approach.

#### Comments

Contents such as integration by partial fractions, integration by substitution, integration by parts are no longer to be found in today's textbooks. Also, many curves can be better represented in parameter form. Years ago, it was possible to address this issue by limiting the number of learning areas to two out of three (calculus, linear algebra, and stochastics) and at the same time increasing the number of hours per week in advanced courses, especially since this form of presentation offers the possibility to repeat and consolidate the classes of functions treated so far

#### Die Fläche von Ellipsen – Area of Ellipses

Betrachtet wird nur der obere rechte Ellipsenbogen mit der Gleichung  $y = \frac{b}{a} \cdot \sqrt{a^2 - x^2}$ .

Dann errechnet sich die Fläche zu  $A = 4 \cdot \frac{b}{a} \cdot \int_0^a \sqrt{a^2 - x^2} dx$ .



Mit der Substitution  $\left. \begin{array}{l} x = a \cdot \sin(z) \\ \frac{dx}{dz} = a \cdot \cos(z) \end{array} \right\}$  und den geänderten Grenzen 0 und  $\frac{\pi}{2}$  erhält man

$$A = \frac{4b}{a} \cdot \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \cdot \sin^2(z)} \cdot a \cdot \cos(z) dz \text{ und vereinfacht: } A = 4 \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \cos^2(z) dz.$$

Das Integral  $\int \cos^2(z) dz$  löst man mittels partieller Integration:  $\int u' \cdot v dz = u \cdot v - \int u \cdot v' dz$ .

Only the upper right elliptic arc is considered with the equation  $y = \frac{b}{a} \cdot \sqrt{a^2 - x^2}$ . This leads to the

$$\text{area } A = 4 \cdot \frac{b}{a} \cdot \int_0^a \sqrt{a^2 - x^2} dx.$$

Using substitution  $\left. \begin{array}{l} x = a \cdot \sin(z) \\ \frac{dx}{dz} = a \cdot \cos(z) \end{array} \right\}$  and changing bounds 0 und  $\frac{\pi}{2}$  we obtain

$$A = \frac{4b}{a} \cdot \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \cdot \sin^2(z)} \cdot a \cdot \cos(z) dz \text{ and simplified: } A = 4 \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \cos^2(z) dz.$$

For the integral  $\int \cos^2(z) dz$  we apply integration by parts:  $\int u' \cdot v dz = u \cdot v - \int u \cdot v' dz.$

$$\int \cos(z) \cdot \cos(z) dz \text{ mit } \begin{array}{ll} u' = \cos(z) & u = \sin(z) \\ v = \cos(z) & v' = -\sin(z) \end{array}$$

Daraus folgt:

$$\left. \begin{array}{l} \int \cos(z) \cdot \cos(z) dz = \sin(z) \cdot \cos(z) + \int \sin^2(z) dz \\ \int \cos(z) \cdot \cos(z) dz = \sin(z) \cdot \cos(z) + \int 1 - \cos^2(z) dz \\ \int \cos(z) \cdot \cos(z) dz = \sin(z) \cdot \cos(z) + z - \int \cos^2(z) dz \\ 2 \cdot \int \cos^2(z) dz = z + \frac{1}{2} \sin(2z) \end{array} \right\} \Rightarrow \int \cos^2(z) dz = \frac{z}{2} + \frac{1}{4} \cdot \sin(2z)$$

*So etwas händisch zu lösen, war im Studium des Maschinenbau Standard!*

*It was "Standard" to perform such calculations by paper and pen during study of mechanical engineering!*

Betrachtet man noch einmal das Ausgangsintegral  $A = 4 \cdot \frac{b}{a} \cdot \int_0^a \sqrt{a^2 - x^2} dx$ , so führt das mit der Substitution auf das Ergebnis

If we look again at the initial integral from above then substitution leads to the final result

$$A = 4 \cdot a \cdot b \cdot \left[ \frac{z}{2} + \frac{1}{4} \cdot \sin(2z) \right]_0^{\frac{\pi}{2}} = a \cdot b \cdot \pi$$

Ersetzt man wiederum  $z$  in  $\frac{z}{2} + \frac{1}{4} \cdot \sin(2z)$  durch  $z = \sin^{-1}\left(\frac{x}{a}\right)$  (siehe Substitution) so erhält man

Substituting for  $z$  in  $\frac{z}{2} + \frac{1}{4} \cdot \sin(2z)$  by  $z = \sin^{-1}\left(\frac{x}{a}\right)$  (see substitution) we obtain

$$\left. \begin{array}{l} \frac{\sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{1}{2} \cdot \sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right) \cdot \cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right) \\ = \frac{1}{2} \cdot \left[ \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a} \cdot \sqrt{1 - \left(\frac{x}{a}\right)^2} \right] \\ = \frac{1}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a^2} \cdot \sqrt{a^2 - x^2} \end{array} \right\} \rightarrow \text{Mit den Grenzen } x = 0 \text{ und } x = a \text{ erhält man}$$

$$A = 4 \cdot a \cdot b \cdot \left[ \frac{1}{2} \sin^{-1} \left( \frac{x}{a} \right) + \frac{x}{a^2} \cdot \sqrt{a^2 - x^2} \right]_0^a =$$

$$= 4 \cdot a \cdot b \cdot \left[ \underbrace{\frac{1}{2} \sin^{-1}(1)}_{=\frac{\pi}{2}} + 0 - \left( \frac{1}{2} \sin^{-1}(0) + 0 \right) \right] = a \cdot b \cdot \pi$$

Ein einfacherer Weg – *An easier way*:

Die (vektorielle) Parameterform von Ellipsen lautet:  $\vec{e}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a \cdot \cos(t) \\ b \cdot \sin(t) \end{pmatrix}$ .

Flächen berechnen sich nach der Formel  $A = \int_{t_1}^{t_2} y(t) \cdot \dot{x}(t) dt$  und man erhält hier:

The parameter form (in vector representation) of ellipses is given by  $\vec{e}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a \cdot \cos(t) \\ b \cdot \sin(t) \end{pmatrix}$  and

areas are calculated using  $A = \int_{t_1}^{t_2} y(t) \cdot \dot{x}(t) dt$ , which gives applied in our case:

$$\left. \begin{aligned} A &= 4 \cdot \int_{-\frac{\pi}{2}}^0 b \cdot \sin(t) \cdot (-a \cdot \sin(t)) dt \\ A &= -4 \cdot a \cdot b \cdot \int_{-\frac{\pi}{2}}^0 \sin^2(t) dt \\ A &= -4 \cdot a \cdot b \cdot \left[ \frac{x}{2} - \frac{1}{2} \sin(2x) \right]_{-\frac{\pi}{2}}^0 \end{aligned} \right\} \rightarrow \underline{A = a \cdot b \cdot \pi}$$

$\int \sin^2(t) dt$  mit partieller Integration wie oben beschrieben! (integration by parts as described above!)

**Haupt- und Nebenachsen / Abstand der Kurven – Major and minor axes / distance of the curves**

Es gilt:  $\left. \begin{aligned} a \cdot b \cdot \pi &= 121 \\ a : b &= 2 : 1 \end{aligned} \right\} \Rightarrow$

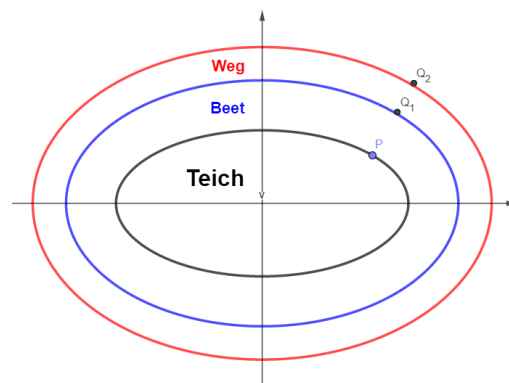
$$\boxed{\begin{aligned} b &= \frac{11 \cdot \sqrt{2\pi}}{2\pi} \\ a &= \frac{11 \cdot \sqrt{2\pi}}{\pi} \end{aligned}}$$

In der Grafik sind zwei weitere Ellipsen gezeichnet, deren Haupt- und Nebenachsen 3m bzw. 5m größer sind als die der Teich-Ellipse.

Man setzt einen Punkt P auf der Teich-Ellipse und errichtet in diesem Punkt die Normale. Im Abstand  $d_1 = 3$  bzw.  $d_2 = 5$  konstruiert man die Punkte  $Q_1$  bzw.  $Q_2$ .

Sichtbar ist, dass diese nicht auf der blauen bzw. roten Ellipse liegen.

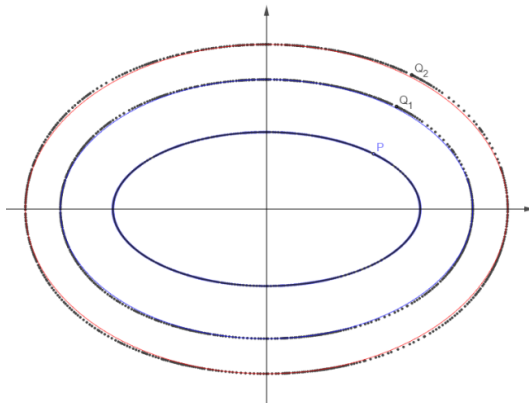
Das Beet bzw. der Weg haben also auf den Ellipsen nicht überall die Breite 3 m bzw. 2m.



In the graphic, two other ellipses are drawn, whose major and minor axes are 3m and 5m larger, respectively, than those of the pond ellipse.

Set a point P on the pond ellipse and construct the normal at this point. At the distance  $d_1 = 3$  or  $d_2 = 5$  you construct the points  $Q_1$  and  $Q_2$  respectively. One can observe that these are not on the blue and red ellipse respectively.

Thus, the bed or path on the ellipses do not have the width of 3m or 2m everywhere.



Verfolgt man die Spurpunkte, ergeben sich aber nur geringe Unterschiede zwischen den Ellipsen und der Spur der Punkte  $Q_1$  und  $Q_2$ .

Die geforderten Abstände liegen daher nicht auf Ellipsen.

Aus all dem ergibt sich:

*Die größte Ellipse hat den Flächeninhalt 121 m<sup>2</sup>*

Die Gleichungen der äquidistanten Kurven sollen nun noch ermittelt werden.

Following the track points, however, there are only slight differences between the ellipses and the track of the points  $Q_1$  and  $Q_2$ .

From all this results: *The largest ellipse has the area 121 m<sup>2</sup>.*

Finally, we'd like to find the equations of the equidistant curves.

### Äquidistante Kurven - Equidistant curves

In der Aufgabenstellung gibt es also nur eine Ellipse und das ist die mit 121 m<sup>2</sup>. Die außen liegenden Kurven weichen leicht von einer Ellipsenform ab.

Hier stellt sich die Frage, wie man zu einer Ellipse der Form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  eine äquidistante Kurve ermittelt. Das Prinzip ist unter Punkt 3) schon annähernd beschrieben und soll nun rechnerisch durchgeführt werden. Dies geschieht einfacher, wenn man die Ellipse in Parameterform (vektoriell) darstellt:

In the problem definition there is only one ellipse and that is the one with 121 m<sup>2</sup>. The curves on the outside deviate slightly from an elliptical shape. Here the question arises how to determine an equi-

distant curve to an ellipse of the form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . This is done more easily if the ellipse is represented in parameter (vector) form.

We use its first derivative, and find the slope vector and the perpendicular vector:

$$\vec{e}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a \cdot \cos(t) \\ b \cdot \sin(t) \end{pmatrix} \quad \text{Vektor vom Ursprung zum Punkt P: } \vec{ur} = \begin{pmatrix} \cos(t) \\ b \cdot \sin(t) \end{pmatrix}$$

Die Ableitung bei Kurven in Parameterform erhält aus  $e'(t) = \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{b}{a} \cdot \cot(t)$

Damit ergibt sich der Steigungsvektor zu

$$\vec{m} = \begin{pmatrix} 1 \\ \frac{b}{a} \cdot \cot(t) \end{pmatrix}$$

und der Normalenvektor ergibt sich zu

$$\vec{n} = \begin{pmatrix} \sin(t) \\ a \end{pmatrix}$$

We need the unit perpendicular vector (length 1) which must be multiplied by the distance of the curves. Then it is no problem to find the vector  $\vec{OQ}$  from the origin to a point of the curve, which describes the upper part of the curve in parameter form (as vector).

Benötigt wird der Normaleneinheitsvektor (Vektor der Länge 1), der dann noch mit der Distanz d der Kurven multipliziert werden muss.

$$\vec{n}_0 = \frac{\vec{n}}{|\vec{n}|} \quad \text{mit} \quad |\vec{n}| = \frac{(t) \cdot (a^2 - b^2) + b}{a \cdot \sin(t)}$$

Der Vektor zum Punkt Q ergibt sich nun zu  $\vec{OQ} = \vec{OR} + a \cdot \frac{\vec{n}}{|\vec{n}|}$ .

Man erhält schließlich den oberen Teil der äquidistanten Kurve in vektorieller Form zu

$$\vec{OQ} = \begin{pmatrix} a \cdot \cos(t) + d \cdot \frac{a \cdot \sin(t)}{\sqrt{\sin^2(t) \cdot (a^2 - b^2) + b^2}} \cdot \frac{b}{a} \cdot \cot(t) \\ b \cdot \sin(t) + d \cdot \frac{a \cdot \sin(t)}{\sqrt{\sin^2(t) \cdot (a^2 - b^2) + b^2}} \end{pmatrix} \quad \text{mit } 0 < t < \pi$$

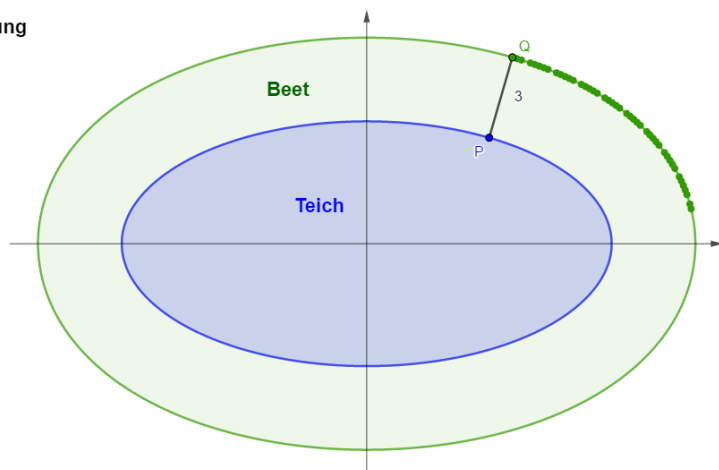
## GeoGebra file

### Eine GeoGebra-Datei zur Veranschaulichung

Die Kurve um den Teich hat die Form einer Ellipse.  
Im Punkt P wurde die Normale errichtet und der Punkt Q im Abstand d = 3 konstruiert.  
Mit der Schaltfläche [start] beginnt die Aufzeichnung der Spur des Punktes Q um die Ellipse.


 Äquidistante zuschalten

### Äquidistante zur Ellipse



#1: 
$$\left( \text{SOLUTIONS} \left[ \int_{-a}^a \frac{b}{a} \cdot \sqrt{(a-x)^2 - x^2} dx = \frac{121}{2} \wedge a = 2 \cdot b, [a, b], \text{Real} \right] \right)_1$$

#2: 
$$\left[ \frac{11 \cdot \sqrt{2}}{\sqrt{\pi}}, \frac{11 \cdot \sqrt{2}}{2 \cdot \sqrt{\pi}} \right]$$

#3:  $e(a, b) := [a \cdot \cos(t), b \cdot \sin(t)]$

#4:  $e\left(\frac{11 \cdot \sqrt{2}}{\sqrt{\pi}}, \frac{11 \cdot \sqrt{2}}{2 \cdot \sqrt{\pi}}\right)$

#5:  $m(a, b) := [-a \cdot \sin(t), b \cdot \cos(t)]$

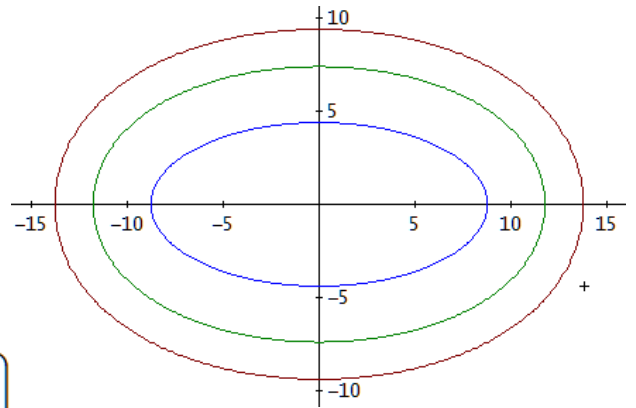
#6:  $n(a, b) := [b \cdot \cos(t), a \cdot \sin(t)]$

#7:  $n0(a, b) := \frac{n(a, b)}{|n(a, b)|}$

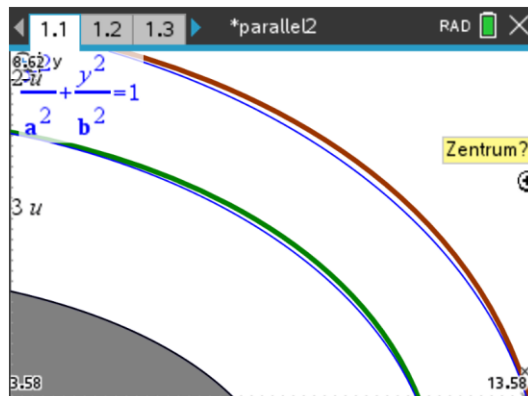
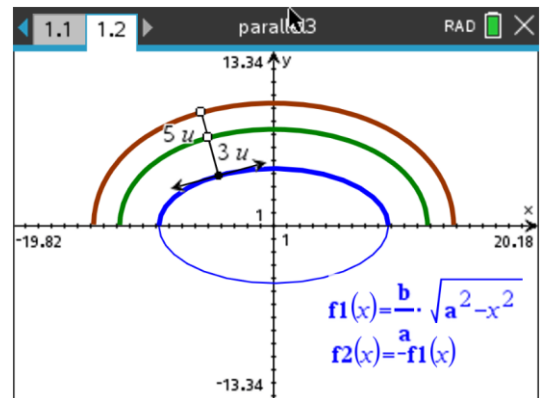
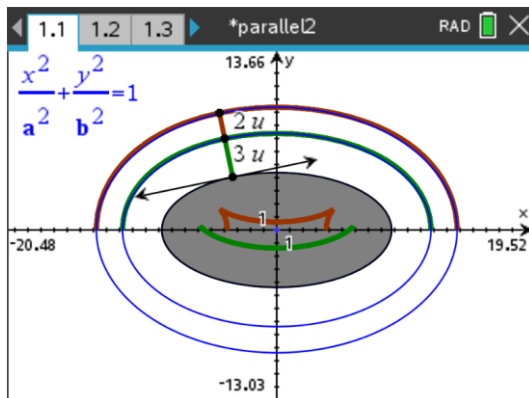
#8:  $e(a, b) + d \cdot n0(a, b)$

#9:  $e\left(\frac{11 \cdot \sqrt{2}}{\sqrt{\pi}}, \frac{11 \cdot \sqrt{2}}{2 \cdot \sqrt{\pi}}\right) + 3 \cdot n0\left(\frac{11 \cdot \sqrt{2}}{\sqrt{\pi}}, \frac{11 \cdot \sqrt{2}}{2 \cdot \sqrt{\pi}}\right)$

#10:  $e\left(\frac{11 \cdot \sqrt{2}}{\sqrt{\pi}}, \frac{11 \cdot \sqrt{2}}{2 \cdot \sqrt{\pi}}\right) + 5 \cdot n0\left(\frac{11 \cdot \sqrt{2}}{\sqrt{\pi}}, \frac{11 \cdot \sqrt{2}}{2 \cdot \sqrt{\pi}}\right)$



Construction of the parallel curves as loci with TI-Nspire. Defining the base ellipse as implicit function does not give the requested curves as a whole (direction of the perpendicular?). Defining the upper and lower half as functions gives upper and lower part of the parallel curves.



## Further Communication with Robert Haas

### Robert's Mail from June 9, 2022

Dear Josef:

Thanks for your draft article about the “Cello Tangents.” You did a *lot* of work; it took me several days to (try to) go through it! My comments, mostly minor, are attached. I suppose we have to list me as an author, because you started by reproducing my article's example and its attractive cello picture. But the big majority of the work is yours, applying your computer algebra system that I know practically nothing about, and I'm feeling pretty uncomfortable here now, “mich mit fremden Federn zu schmücken”. Can we please at least put you as the first author?

P.S.: Attached are two more items, one quite old, one quite new, that might perhaps suggest topics for your *Newsletter*. The old one is from the program notes I wrote for my Bach recital thirty years ago (my main instrument is actually the violin), where, playing the first prelude from the Well-Tempered Clavier, I delved into “tempering” scales. Mathematically, the problem is finding rational approximations for an irrational number ( $\log 2 / \log 3$ ), a deep question I've never studied (see Hardy & Wright, *An Introduction to the Theory of Numbers*, 5th edition, Chapter XI). But I had fun explaining how one would get a much better approximation if the piano keyboard had 53 notes per octave, rather than the conventional 12.

The other item is a problem I'd sent to *Mathematics Magazine* in 2020, with the solution--the editor simplifying my original problem statement at the cost of adding five extra trivial solutions--published just a few months ago. I was surprised how complicated this simple situation became, arising from just four points in the plane. But I never studied geometry beyond my sophomore year in high school. The published solution by the evidently more experienced reader pointed out a fact I'd missed, that the configuration is actually four of the five points of a regular pentagon. Do you know of any computer programs that will sort out geometric configurations like this that consist of just a few points?

Dear Robert,

Many thanks for your mail full of exciting attachments.

1. Thanks for proof reading.
2. I do not intend to change the order of the authorship. It is my strong opinion that creating a problem is much more important than adding comments, remarks, additions and sometimes even improvements.
3. I enjoyed your questions and faced the challenges (see attached pdf-file).
4. Also, thanks for your papers. I must admit that I had not time until now to study them thoroughly. I am quite sure that they are very suitable for our journal. I don't know computer programs that will sort out geometric configurations.

I'd like to include your so valuable comments in our next newsletter, if you don't mind, of course.



**Robert:**

**COMMENTS ON YOUR MAY 30 “CELLO TANGENTS”**

p. 23 line 1: I’d suggest replacing “Inspired by a picture of Philip Choi” with “Seeing the bow resting on top of the cello (photo).”

The actual inspiration was, of course, my *own* cello--looking at it while my brain was in “math mode,” and realizing that the bow resting on top was like a line tangent to a curve at two points. Philip Choi’s picture of his cello came about only at the very end, through intervention by *J. Humanistic Math.* editor Gizem Karaali. She emailed me about a week before publication to say that my article--at that point containing just the math diagram--would be much more attractive if we would add a picture of my cello + bow. But alas I didn’t own a digital camera to take a photo; we hunted for a while online, but all the cello pictures were from the wrong angle, and/or very expensive. With publication just a few days away, Prof. Karaali remembered that her collaborator, astronomy-physics Prof. Choi (Karaali, Choi, Sood, and Grosfils, “Envisioning a quantitative studies center: A liberal arts perspective,” *Numeracy* 3(1) (2010) article 4, pp. 1-18) was an amateur cellist, and she was able to get him to take the lovely photo now adorning my article. It takes a wise editor, both to see like that what’s missing, and to know the right people to push at the last minute to correct the problem!

p. 23: §3 lines 2-3 (“What is the equation of the profile of the instrument?”) and p. 25: §1 (“...reverse the task. Robert reconstructed the profile ...”)

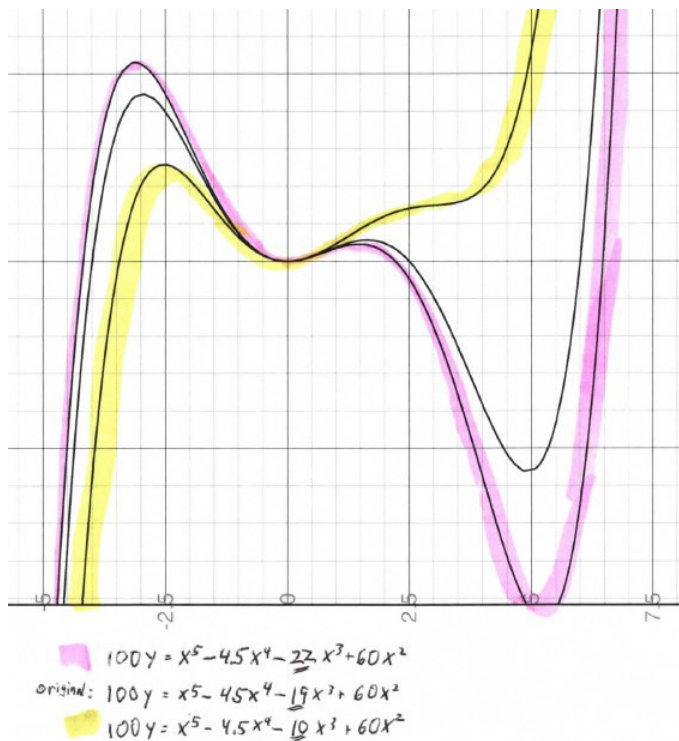
I think my own work was much more limited and specialized: Recognizing that the cello + bow suggested a double tangent problem, I *constructed an example* of a double tangent problem simple enough to solve analytically by matrix inversion. That the bow rests on top (i.e. the line is tangent to the curve at two points) is far too little information to “reconstruct” the “equation of the profile of the instrument”- it merely gives one constraint, and I was able to construct an example satisfying that constraint.

I think your own work is much more general, and does not “reverse” mine: you are able, by Groebner Basis computer algebra, to *solve the double tangent problems* for your (arbitrarily chosen?) 4th, 5th, and 6th-degree polynomials.

pp. 26-30 Quintic:

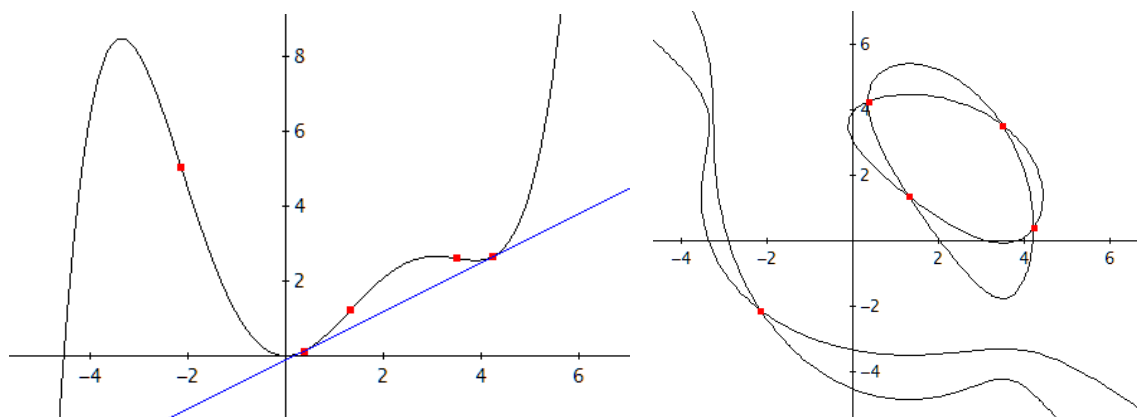
I’ve never studied algebraic geometry--do you know whether there’s some general theory (maybe “genera of curves”?) allowing one to predict their detailed shape, specifically whether a “cello tangent” even *exists*? I played around with the coefficients for  $x^4$ ,  $x^3$ , and  $x^2$  in your example here,  $z(x) = 0.01x^5 - 0.045x^4 - 0.19x^3 + 0.6x^2$ , raising (pink) or lowering (green) their absolute values compared to the original ones (unshaded) in the attached plots, and in each case I could find a value  $-.045 \rightarrow .02$ ,  $.19 \rightarrow .1$ , or  $.6 \rightarrow 1.0$  – with *no* cello tangent. Does something notable happen in your implicit function plot in such situations?

Next page shows one of Robert’s plots.



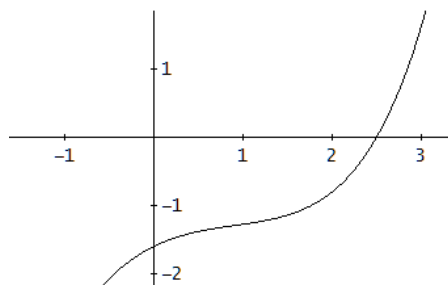
My (Josef) response:

I believe that the yellow curve enables a “cello tangent”, too (in mathematical sense, of course). Or, do you accept “cello tangents” only then if they are resting above the cello?

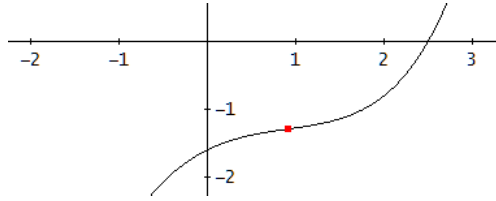


The 5<sup>th</sup> order polynomial shown below cannot give a “cello tangent”.

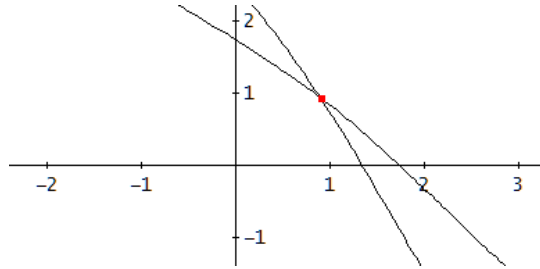
$$zz(x) := \frac{(x - 2.5) \cdot (x^2 + 4) \cdot (x^2 + 16)}{100}$$



Calculation presents the point of inflection as the only one solution.



And this is the respective implicit plot with only one intersection point.



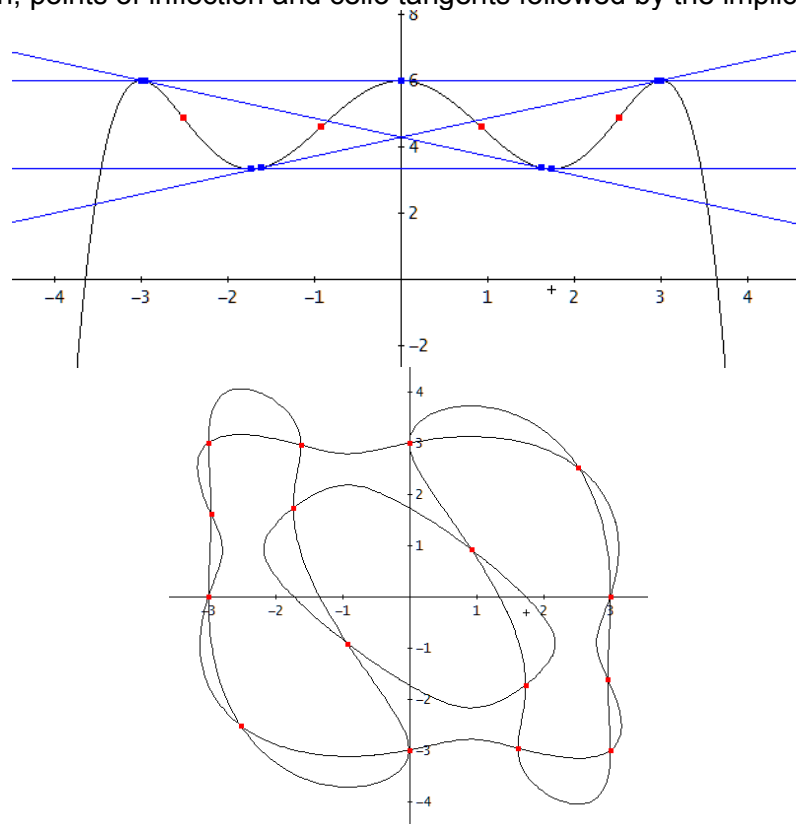
DERIVE file Ce11o\_add.dfw

p. 30, last figure: Can you construct an example of a 6th degree polynomial with *three* “cello tangents” on one line?

Yes, I can: take function  $ff(x)$ :

$$ff(x) := - \frac{2 \cdot (x^6 - 18 \cdot x^4 + 81 \cdot x^2 - 243)}{81}$$

See the function, points of inflection and cello tangents followed by the implicit plots:



DERIVE file Ce11o\_add2.dfw

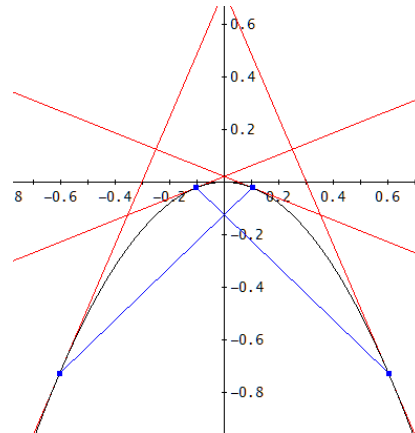
p. 30+: The “cello tangent” problem you just solved seeks pairs of points on a polynomial curve where the tangent lines have the same slopes and the same intercepts. A similar/dual problem might seek point pairs a fixed distance apart (say 1), where the two tangent lines are *perpendicular* (i.e. the product of their slopes is -1). One could call this a “hat problem,” since it’s like trying to perch a hat with a right-angled conical top onto the curve so that it’s tangent at both sides. For example, the curve  $y = -2x^2$  has  $(x_1, x_2) = ((-2+\sqrt{3})/4, (2+\sqrt{3})/4) \approx (-.067, .933)$  as one such pair. Your computer algebra system can doubtless produce higher degree examples.

$$f(x) := -2 \cdot x^2$$

My calculations give two pairs of points with distance 1 and forming a “hat”.

$$\frac{\sqrt{2}}{4} + \frac{1}{4} \qquad \frac{1}{4} - \frac{\sqrt{2}}{4}$$

$$\frac{\sqrt{2}}{4} - \frac{1}{4} \qquad -\frac{\sqrt{2}}{4} - \frac{1}{4}$$



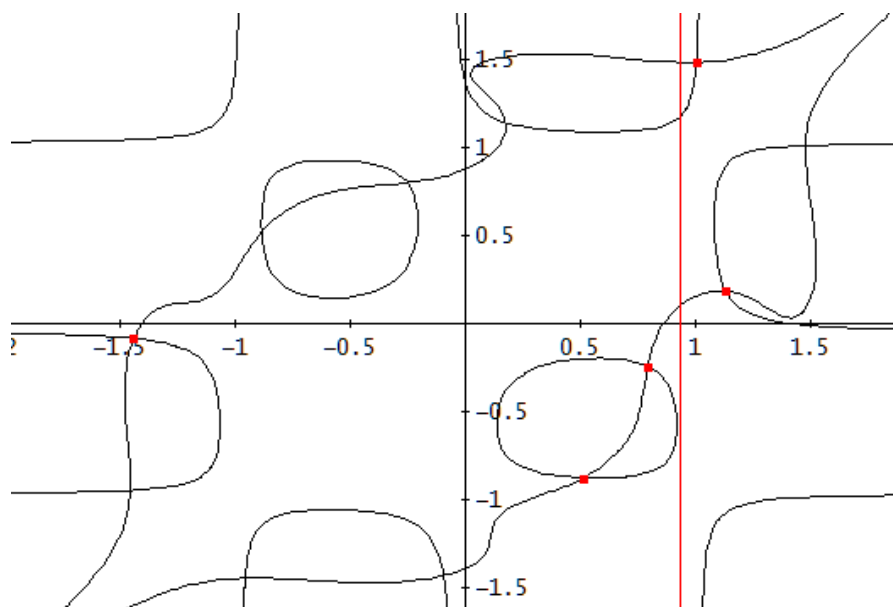
Then I tried the same with a 4<sup>th</sup> order polynomial (with distance 2 between the points):

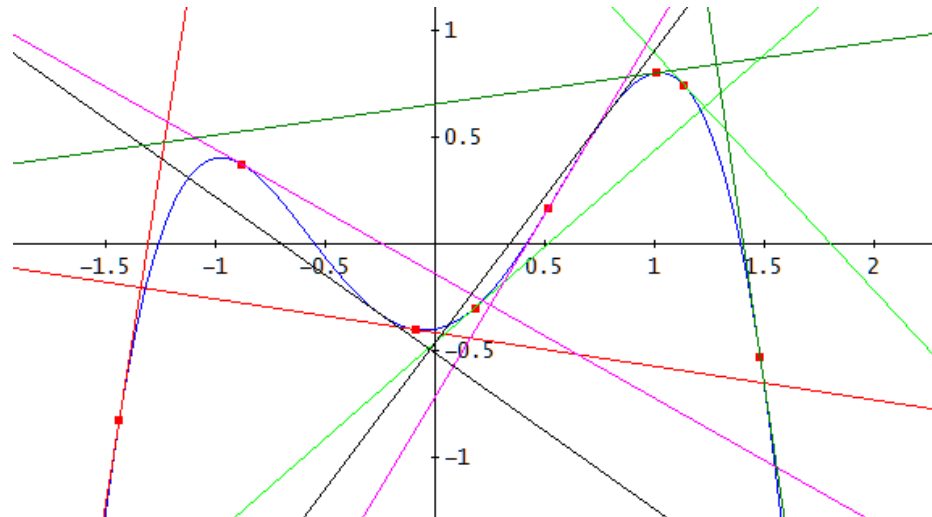
$$f_-(x) := -x^4 + 2 \cdot x^2 + \frac{1}{5} \cdot x - \frac{2}{5}$$

Which gave a nice implicit plot (see below):

Calculation was not so “nice”. Groebner Basis led to an equation of order 48, which had finally 7 real solutions.

Five of them gave intersection points – at least for me –, which correspond with “hats” for the curve (plotted in different colors):



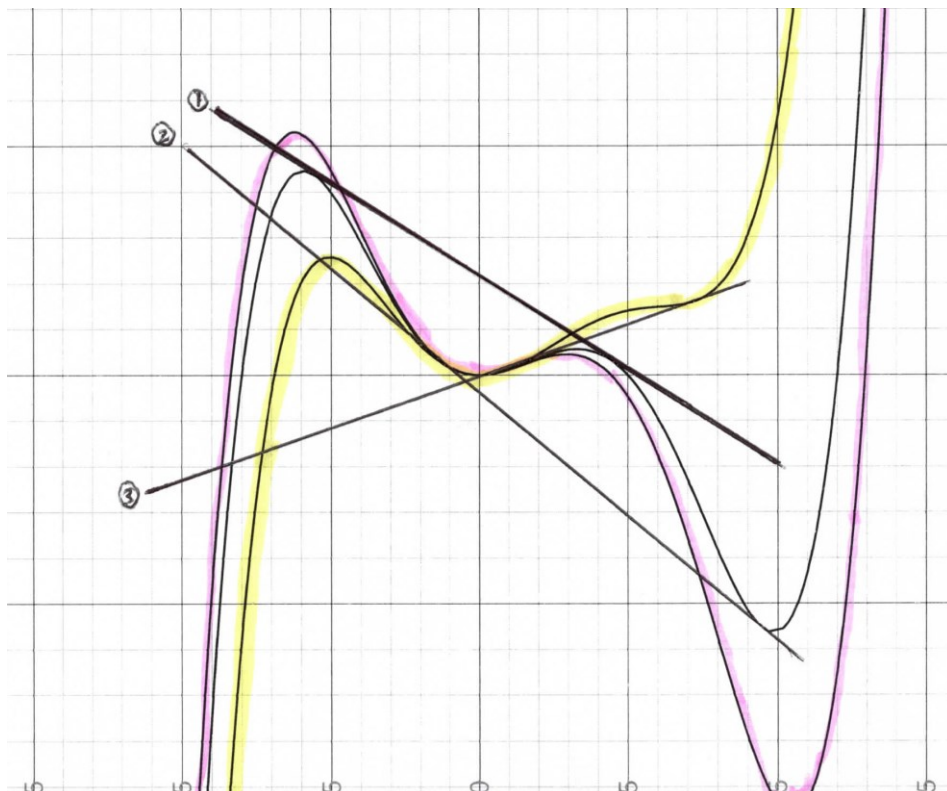


DERIVE file hat.dfw

Dear Josef:

Thanks for the update, and my apologies for my confusion. I had taken so long to get through your unfamiliar material that, rushing then to answer you, I made some conceptual slips:

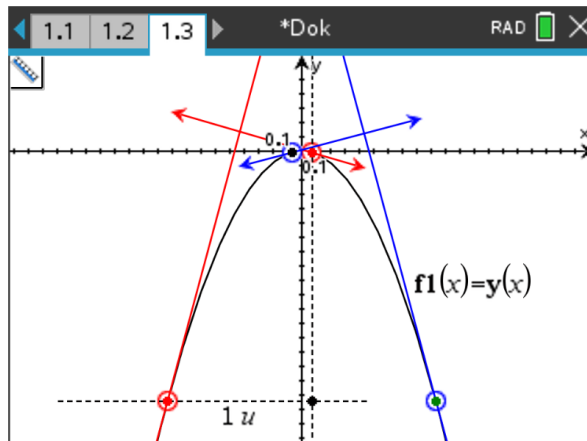
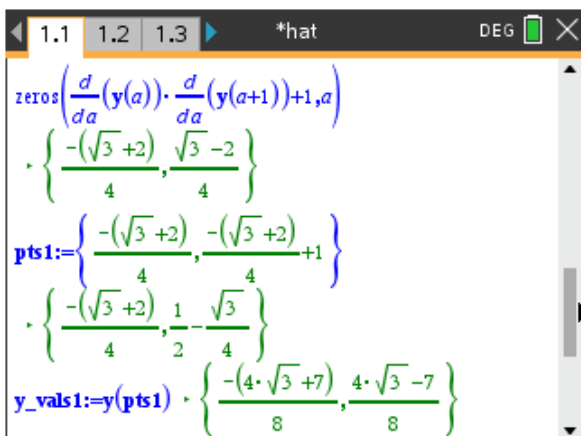
**Cello tangents:** I was, as you surmised, thinking of them as always coming from above like a cello bow, a restriction that, as you point out, isn't mathematically necessary. But using your more general notion of cello tangents then rephrases but doesn't solve my basic question: The original (unshaded) curve in the attached figure has now *two* cello tangents, #1 from above, #2 from below, while the yellow curve has only *one*, #3 from below. Can one predict / explain / understand why?



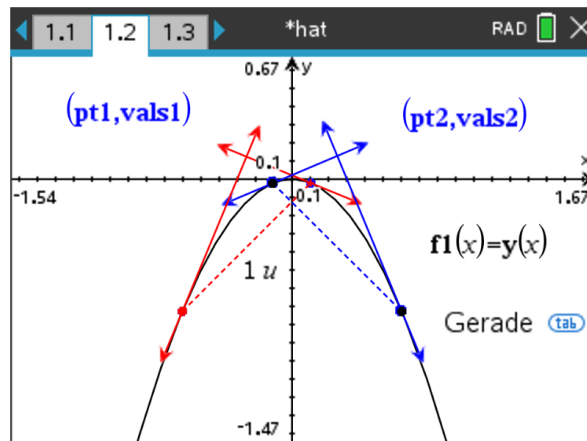
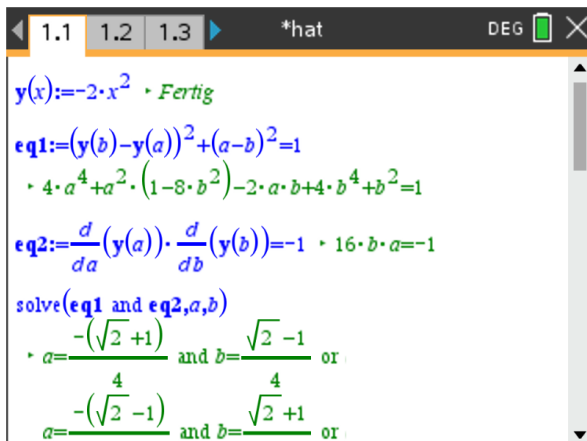
**Hat problem:** To select my pair of points  $(x_1, y_1)$  and  $(x_2, y_2)$  on the parabola  $y = -2x^2$  as end-points of the “hat,” I made life easy for myself *but neglected to explain to you* that I simply chose  $x_2 = 1+x_1$  (thus separating  $x_1$  and  $x_2$  by a distance of 1), yielding my solution  $x_1 = (-2+\sqrt{3})/4 \approx -.067$ ,  $y_1 = -2(x_1)^2$ ,  $x_2 = 2+\sqrt{3})/4 \approx .933$ ,  $y_2 = -2(x_2)^2$ . I believe you constrained your two points  $(x_1, y_1)$  and  $(x_2, y_2)$  to be the distance 1 apart (what I *said*, not what I *meant to say!*), a harder, messier problem. My apologies!

Robert

This, what Robert had in mind (performed on TI-NspireCAS:



Followed by my interpretation of the distance:



Finally, I tried to solve Robert’s problem with my (slightly changed) 4<sup>th</sup> order polynomial.

I got four pairs of points with horizontal distance 2 creating a “hat”. There are only two pairs which can be displayed on the screen.

Calculation and solution are given in the file.

TI-Nspire file hat.tns

