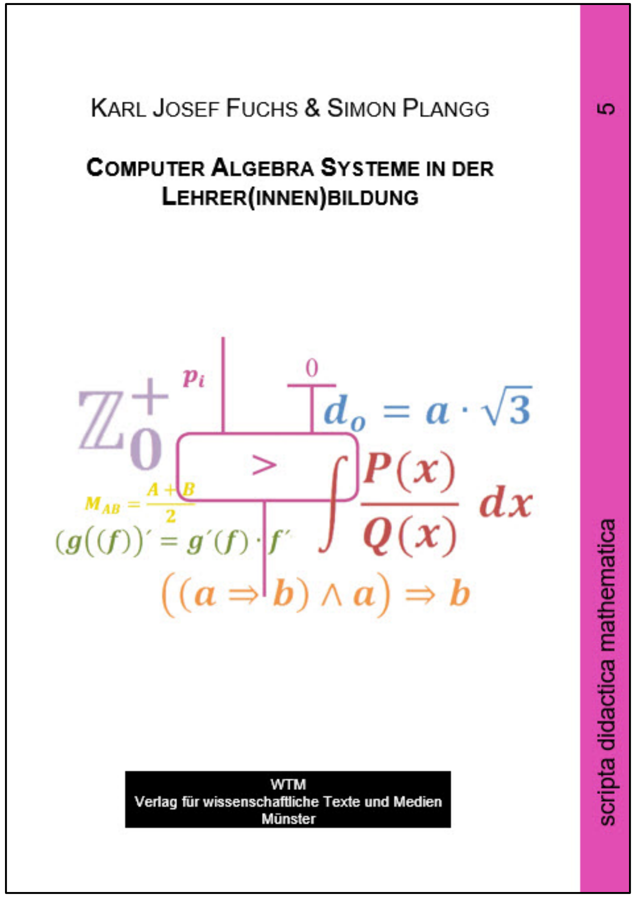
Symbolic Computation in Teacher Education

Simon Plangg

PÄDAGOGISCHE HOCHSCHULE SALZBURG
Stefan Zweig

Technology in mathematics education

- Courses
  - University of Salzburg
  - College of Teacher Education Salzburg
- Prospective math teachers for secondary education
  $\approx$ 60 to 100 per cohort
- Technology in math classes I + II
- CAS in math classes

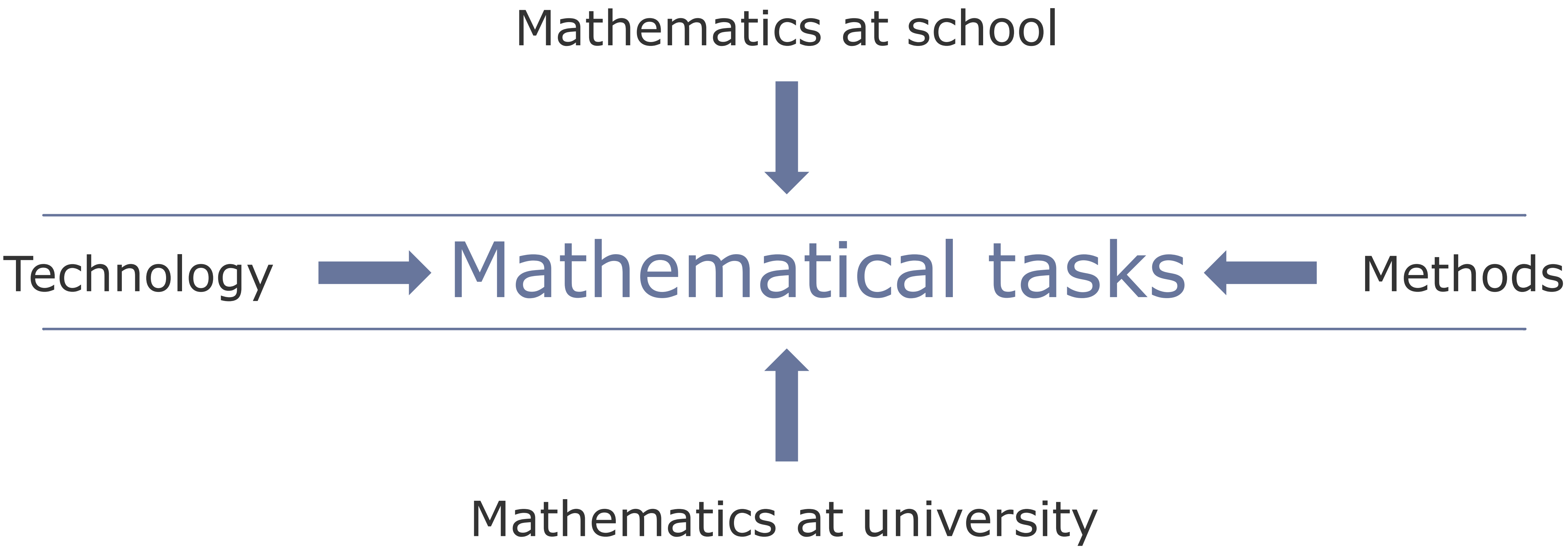


PÄDAGOGISCHE HOCHSCHULE SALZBURG
Stefan Zweig

Content

1. Basic design of the courses
2. Methodological concepts for teaching considering CAS
   - Genetic principle as a fundament
     - Problem-oriented teaching
     - Application-oriented teaching
   - Alignment to fundamental ideas – modeling - programming
   - Illustrating
3. Résumé

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig



PÄDAGOGISCHE HOCHSCHULE SALZBURG

Fundament: Genetic Principle

- Thoughts are integrated in major contexts in- and outside mathematics
  - Problem-oriented teaching
  - Application-oriented teaching
- Intuitive and heuristic approaches lead to more abstract considerations
  - Starting point: inherent understanding
  - Spiral principle

PÄDAGOGISCHE HOCHSCHULE SALZBURG

Wittmann 1981, Fuchs & Plangg 2018

Problem-oriented teaching

- Starting point of knowledge acquisition: problem
  - Intellectual and emotional identification
  - Transfer of knowledge and linking knowledge
- Self-regulated learning
- Problem-based-learning
  - Externalization
  - Contemplation
  - Argumentation

Fuchs & Plangg 2018, Zimmermann 1991, Weber 2007

Externalization

- Task: Find a formula to determine the expectancy value of a binomial distributed random variable
- Major context: dealing with uncertainty
- Design, describe the product of the problem-solving process
- CAS: Formulate

$$e(n):=\sum_{k=0}^{n}\left(k\cdot \mathrm{nCr}(n,k)\cdot p^{k}\cdot (1-p)^{n-k}\right) \qquad \textit{Fertig}$$

$e(n)$

$$\sum_{k=0}^{n}\left(\frac{k\cdot(-(p-1))^{n}\cdot \mathbf{e}^{k\cdot(\ln(p)-\ln(-(p-1)))}}{k!\cdot(n-k)!}\right)\cdot n!$$

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

Externalisation - Contemplation

- Develope, examine possible solutions
- CAS: experimental, heuristic approach (create a great number of examples, outsourcing calculations)
  - Substitute
  - Functional thinking
  - Recognition of patterns
  - Abstract, formulate
  - Discovery learning

$e(n)$	$\sum_{k=0}^{n}\left(\frac{k\cdot(-(p-1))^{n}\cdot \mathbf{e}^{k\cdot(\ln(p)-\ln(-(p-1)))}}{k!\cdot(n-k)!}\right)\cdot n!$
$e(0)$	0
$e(1)$	p
$e(2)$	$2\cdot p$
$e(3)$	$3\cdot p$
$e(4)$	$4\cdot p$
$e(5)$	$5\cdot p$
$e(427)$	$427\cdot p$

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

Contemplation - Evaluation

- Argue, reason a certain solution
- Transfer and link knowledge
  - Binomial theorem
  - Differentiation
- CAS: focus on problem-solving
  - Formulate
  - Manipulate (Substitute)
  - Interpret (Compare)

$$(a+b)^n=\sum_{k=0}\left(\mathrm{nCr}(n,k)\cdot a^k\cdot b^{n-k}\right) \qquad (a+b)^n=\sum_{k=0}\left(\frac{b^{n-k}\cdot a^k}{k!\cdot(n-k)!}\right)\cdot n!$$

$$\frac{d}{da}\left((a+b)^n=\sum_{k=0}^{n}\left(\mathrm{nCr}(n,k)\cdot a^k\cdot b^{n-k}\right)\right) \qquad (a+b)^{n-1}\cdot n=\frac{\sum_{k=0}^{n}\left(\frac{b^{n-k}\cdot a^k\cdot k}{k!\cdot(n-k)!}\right)\cdot n!}{a}$$

$$\left((a+b)^{n-1}\cdot n=\frac{\sum_{k=0}^{n}\left(\frac{b^{n-k}\cdot a^k\cdot k}{k!\cdot(n-k)!}\right)\cdot n!}{a}\right)\cdot a \qquad a\cdot(a+b)^{n-1}\cdot n=\sum_{k=0}^{n}\left(\frac{b^{n-k}\cdot a^k\cdot k}{k!\cdot(n-k)!}\right)\cdot n!$$

$$a\cdot(a+b)^{n-1}\cdot n=\sum_{k=0}^{n}\left(\frac{b^{n-k}\cdot a^k\cdot k}{k!\cdot(n-k)!}\right)\cdot n!\,|\,a=p \text{ and } b=1-p \qquad n\cdot p=\sum_{k=0}^{n}\left(\frac{k\cdot(-(p-1))^n\cdot e^{k\cdot(\ln(p)-\ln(-(p-1)))}}{k!\cdot(n-k)!}\right)\cdot n!$$

$$e \qquad \sum_{k=0}^{n}\left(\frac{k\cdot(-(p-1))^n\cdot e^{k\cdot(\ln(p)-\ln(-(p-1)))}}{k!\cdot(n-k)!}\right)\cdot n!$$

Profke 1994, Fuchs & Plangg 2018

Fundamental ideas

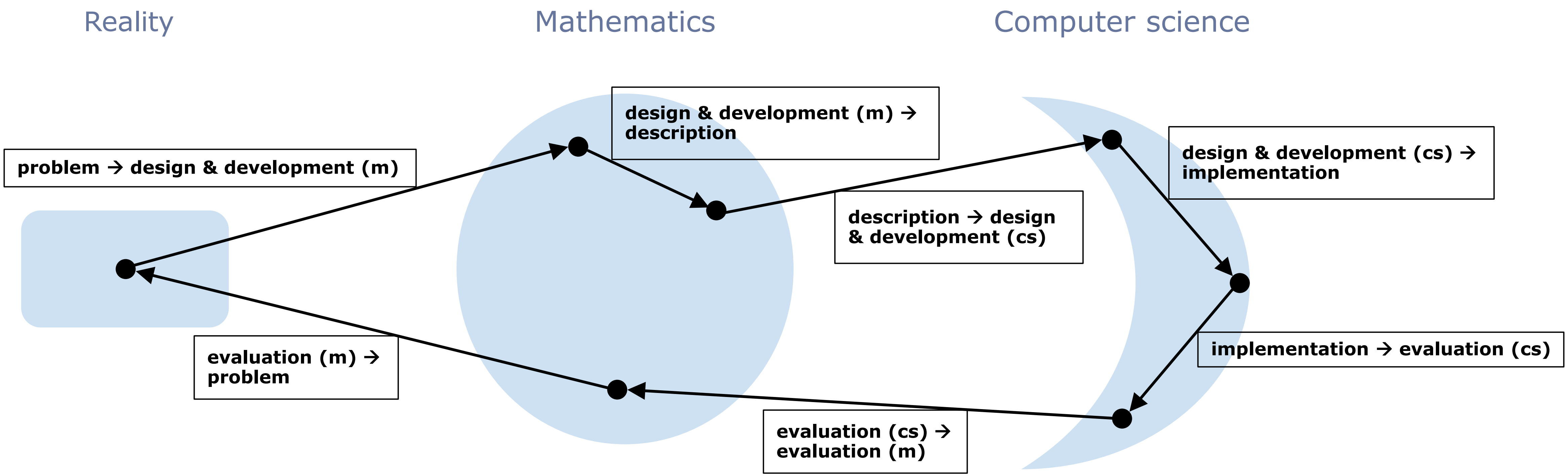
Bundle of actions, strategies and techniques that

- can be identified within the historical development of mathematics
- appears viable structuring curricula vertically
- allows to get an idea what mathematics is about
- helps making instructions both more flexible and more transparent
- is anchored in everyday language and activities

Schweiger 2010

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

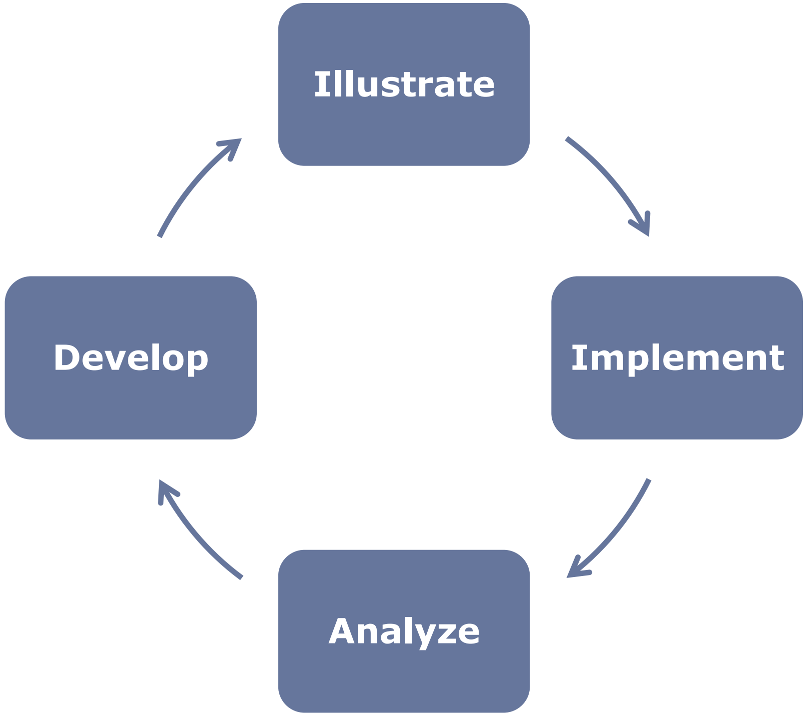
Modeling



Siller 2009

Programming with TI-Nspire CX CAS

- Teaching goal: executable program
- Prototypical approach
  - Authentic problem
  - Choice of programming paradigm
  - Steps of algorithmic thinking



Hubwieser 2007, Kortenkamp & Lambert 2015, Fuchs 1988

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

Prototypical approach

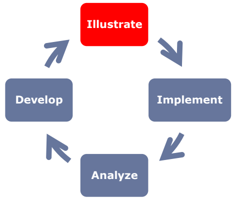
- Problem: How can we (does the Calculator) find approximate values for roots?
- Program editor of the TI-Nspire CX CAS
- Choice of programming paradigm
  - Procedural programming paradigm
    - Sequence of statements
    - Control structures: sequence, conditional branch, repitition
    - TI-Nspire: executable in calculator
  - Functional programming paradigm
    - Language elements: functions
    - Composition of functions
    - TI-Nspire: executable in calculator, graphs, lists and spreadsheet

Nested intervals: „guess and test“

- Task: Estimate $\sqrt{5}$ providing lower and upper bounds with at least three decimals
- What is behind the symbol $\sqrt{5}$?

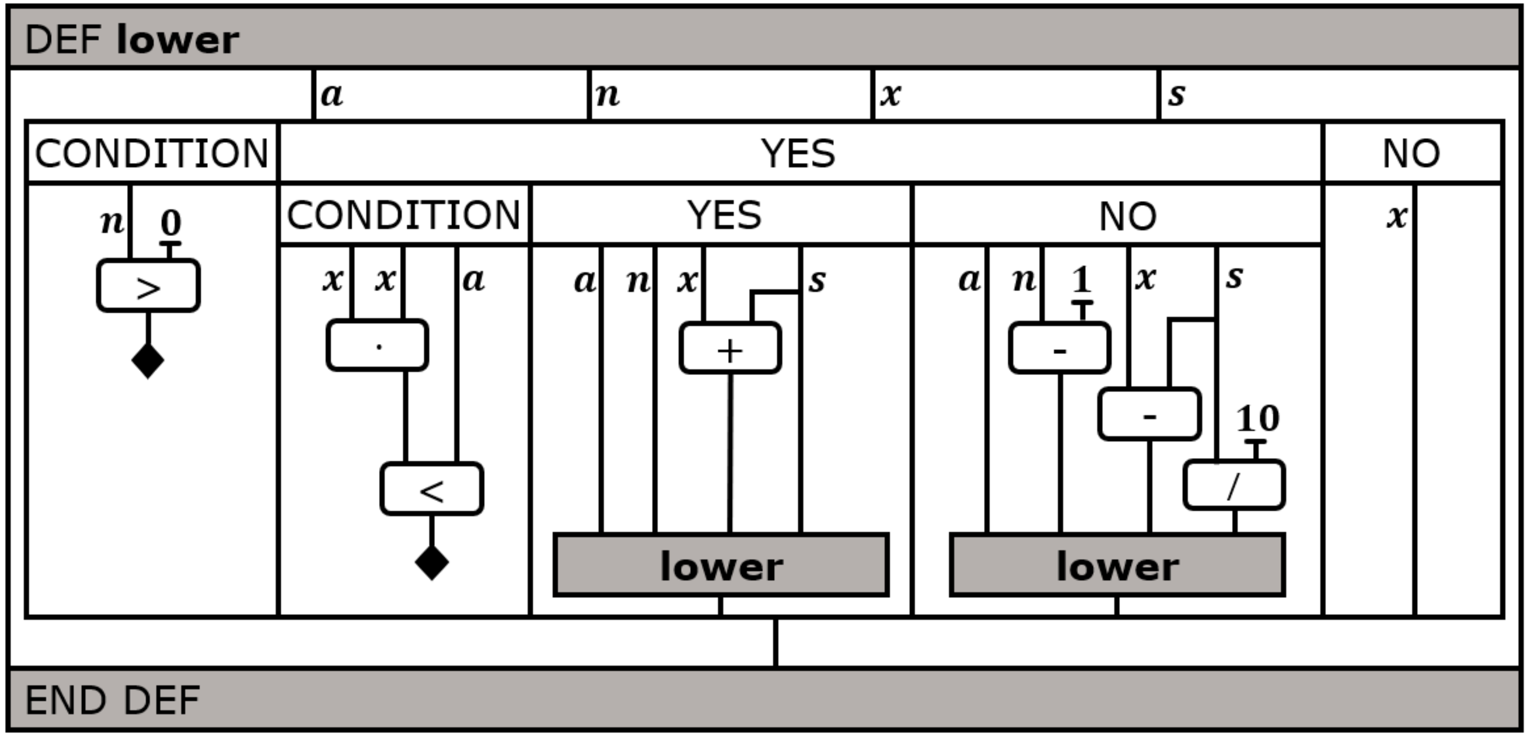
Lower bound	Upper bound
2	3
2.2	2.3
2.23	2.24
2.236	2.237
…	…

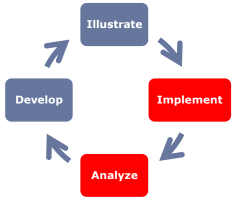
- What is $\sqrt{5}+\pi$ or $2^{\sqrt{5}}$?



PROGRAPH-diagram

- Further possibilities
  - Verbal description
  - Flow chart
  - Structure chart
  - Pseudo code





Nested intervals

- Analyze: Possibilities and limitations of algorithms
- Efficiency

lower 9/9

```
Define lower(a,n,x,s)=
Func
If n>0 Then
 If x^2<a Then
  lower(a,n,x+s,s)
  Else
  lower(a,n-1,x-s,s/10)
 EndIf
 Else
 x
EndIf
EndFunc
```

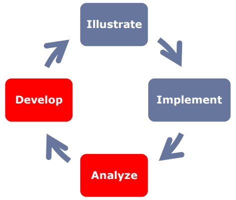
	A n	B lbound	C ubound
=	=seq(i,i,1,1		
1	1	2.2	2.3
2	2	2.23	2.24
3	3	2.236	2.237
4	4	2.236	2.2361
5	5	2.23606	2.23607
6	6	2.236067	2.236068
7	7	2.2360679	2.236068
8	8	2.23606797	2.23606798
9	9	2.236067977	2.236067978
10	10	2.2360679774	2.2360679775

B1 $=\textbf{lower}(5.,a1,1.,10^{-1})$

$lower(1234,10,1.,1)$	35.12833614
$\sqrt{1234}$	35.1283361405
$lower(1234567890,10,1.,1000)$	35136.418286
$\sqrt{1234567890}$	35136.4182864

$\sqrt{5}$	2.2360679775
2.2360679774998	2.2360679775
$lower(1234,3,1.,10^{-1})$	"Fehler: Rekursion zu tief"
$lower(1234,3,30.,10^{-1})$	35.128

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig



Bisection method

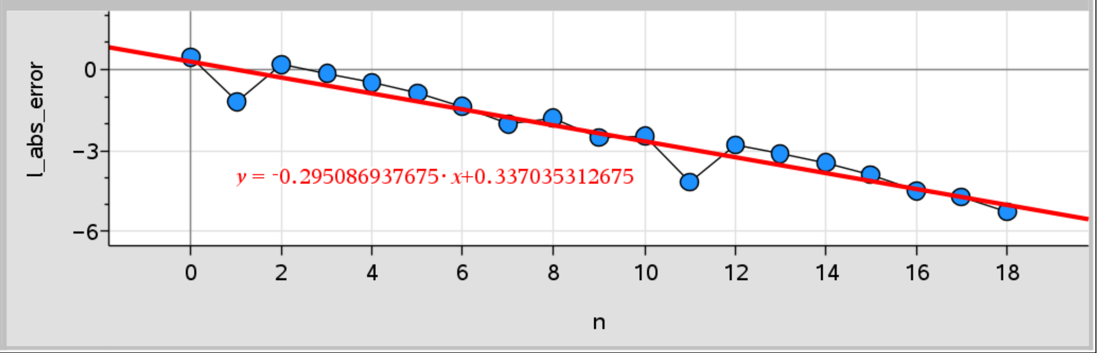
– Possible next step: Heron's method

bisec 8/9

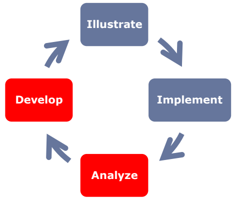
Define **bisec**$(a,lb,ub,n)=$
Func
If $n>0$ Then
  If $\left(\frac{lb+ub}{2}\right)^2<a$ Then
    bisec$\left(a,\frac{lb+ub}{2},ub,n-1\right)$
  Else
    bisec$\left(a,lb,\frac{lb+ub}{2},n-1\right)$
  EndIf
Else
$\begin{bmatrix} lb & ub & \frac{lb+ub}{2} \end{bmatrix}$
EndIf
EndFunc

	A n	B lbound	C ubound	D estimate	E l_abs_error
=	=seq(i,i,0,1				=log(abs(estimate^2−5))
1	0	0.	3.	1.5	0.43933269383
2	1	1.5	3.	2.25	-1.20411998266
3	2	1.5	2.25	1.875	0.171543631305
4	3	1.875	2.25	2.0625	-0.127206598064
5	4	2.0625	2.25	2.15625	-0.455205508061
6	5	2.15625	2.25	2.203125	-0.834933125578
7	6	2.203125	2.25	2.2265625	-1.37243513471
8	7	2.2265625	2.25	2.23828125	-2.00423523382
9	8	2.2265625	2.23828125	2.232421875	-1.78801035052

D9 =**bisec**$(5,b8,c8,1)[1\ \ 3]$



PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig



Heron`s method

– Possible next steps: Cubic root, $\sqrt[n]{\ }$

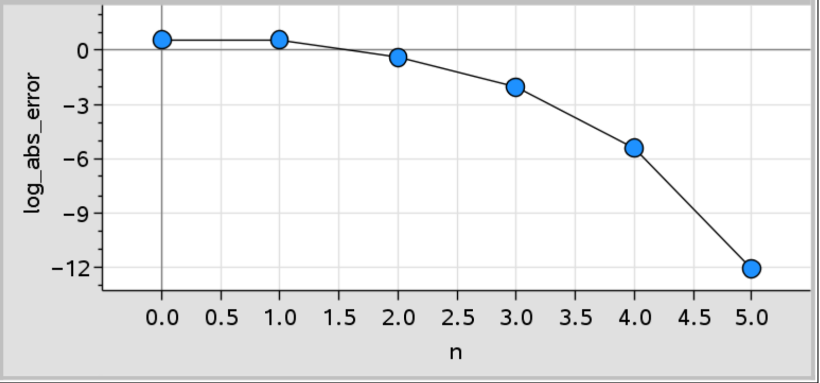
heron 2/5

```
Define heron(a,x,n)=
Func
If n>0 Then
 heron(a,0.5·(x+a/x),n−1)
 Else
 x
EndIf
EndFunc
```

	A n	B estimate	C log_abs_error
=	=seq(i,i,0,7)		=log(abs(estimate^2−5))
1	0	1	0.602059991328
2	1	3.	0.602059991328
3	2	2.33333333333	-0.352182518111
4	3	2.2380952381	-2.04237859812
5	4	2.23606889564	-5.38657429627
6	5	2.2360679775	-12.0457574906
7	6	2.2360679775	#UNDEF
8	7	2.2360679775	#UNDEF

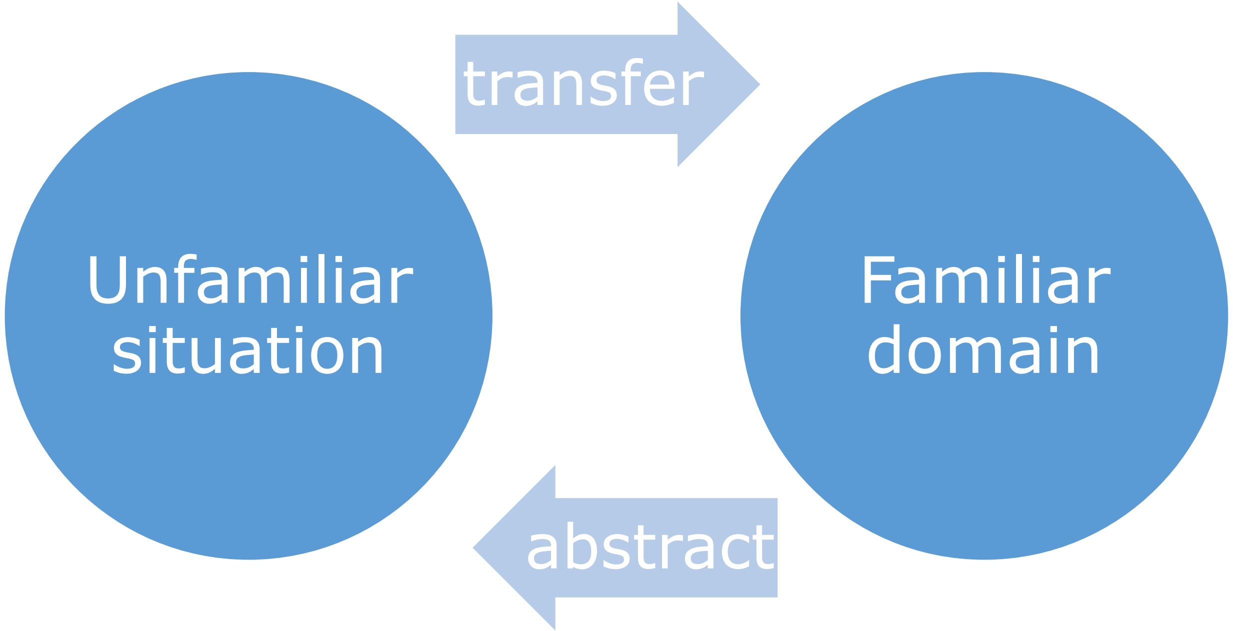
B1 =heron(5,1,a1)

$\sqrt{5}$	2.2360679775
heron(5,1,10)	2.2360679775

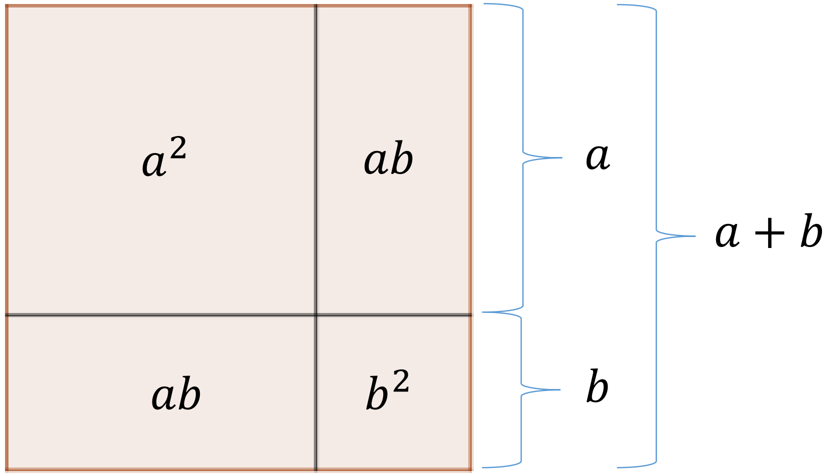


PÄDAGOGISCHE HOCHSCHULE SALZBURG
Stefan Zweig

Illustrating with CAS



$$(a+b)^2 = a^2 + 2ab + b^2$$



- Provide insight in mathematical concepts
- Illustrating is not just visualisation: through geometry, everyday experience, examples, analogy, prove

PÄDAGOGISCHE HOCHSCHULE SALZBURG

Profke 1994

Logics with TI-Nspire CX CAS

- Task: Prove that $\sqrt{2}$ is irrational!
  Illustrate applied inference rules with the TI-Nspire CX CAS!
- Inference rule: proof by contradiction

$\sqrt{2}=\frac{p}{q}$	$\sqrt{2}=\frac{p}{q}$
$\left(\sqrt{2}=\frac{p}{q}\right)^2$	$2=\frac{p^2}{q^2}$
$\left(2=\frac{p^2}{q^2}\right)\cdot q^2$	$2\cdot q^2=p^2$
$2\cdot q^2=p^2 \mid p=2\cdot k$	$2\cdot q^2=4\cdot k^2$
$\frac{2\cdot q^2=4\cdot k^2}{2}$	$q^2=2\cdot k^2$

	A a	B b	C aib	D ainb	E	F na	G reductio_aa
=			='a ⇒ 'b	='a ⇒ not 'b	=aib and ainb	=(not 'a)	=aib and ainb ⇒ na
1	true	true	true	false	false	false	true
2	true	false	false	true	false	false	true
3	false	true	true	true	true	true	true
4	false	false	true	true	true	true	true

a: $\sqrt{2}$ is rational

b: $\sqrt{2}=\frac{p}{q}$ (fraction entirely canceled)

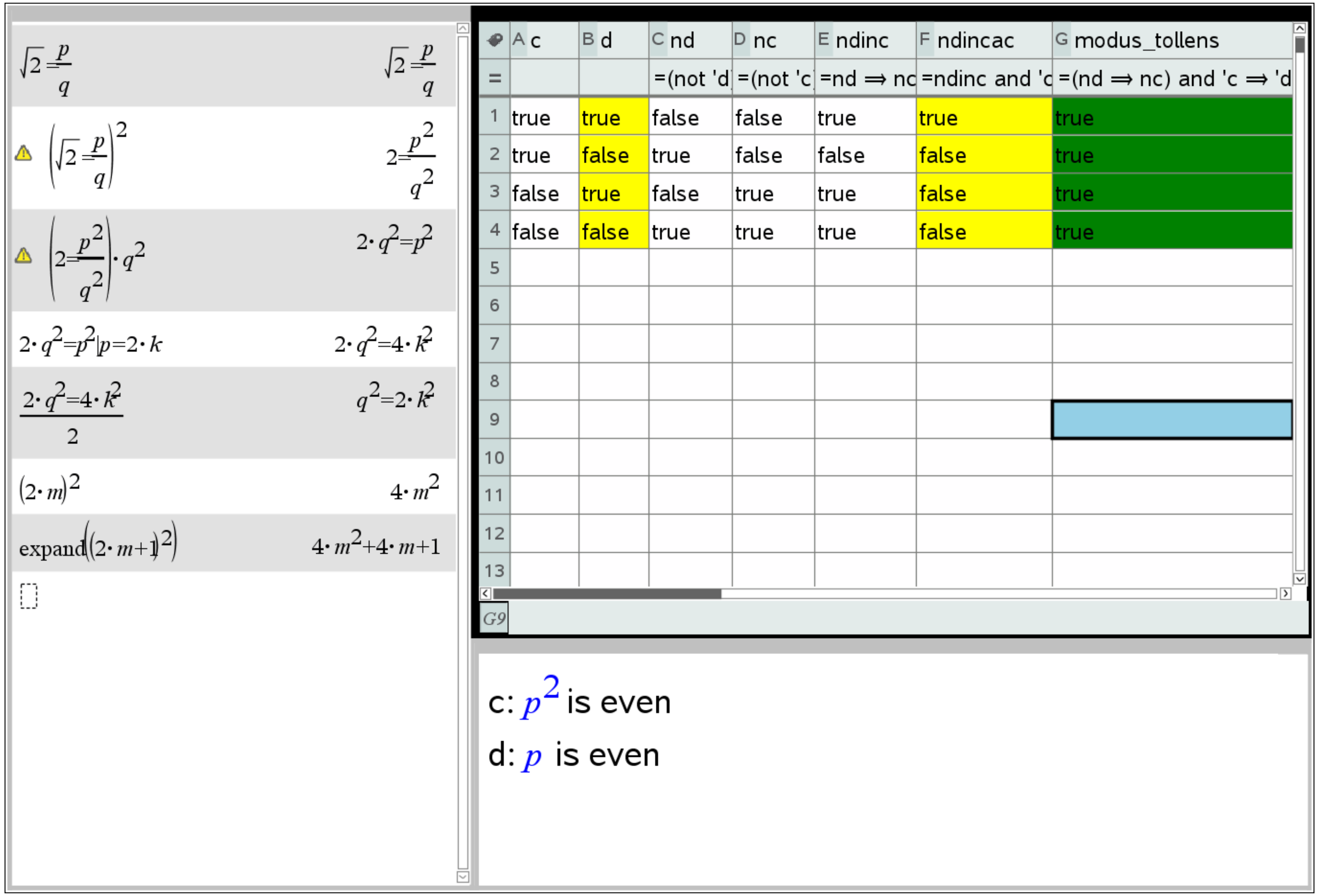
PÄDAGOGISCHE HOCHSCHULE SALZBURG

Logics with TI-Nspire CX CAS

– Modus tollens

$(A \Longrightarrow B) \wedge \neg B \Longrightarrow \neg A$

$(\neg d \Longrightarrow \neg c) \wedge c \Longrightarrow d$



PÄDAGOGISCHE HOCHSCHULE SALZBURG
Stefan Zweig

Application-oriented teaching

- Understanding relevant aspects of our environment
- Method of interdisciplinary teaching: content of other subjects is integrated in (math) classes
- Subject combination: mathematics with geography and economics
  - Geography and economics: demand function, supply function, market equilibrium, equilibrium quantity, Cobweb Theory
  - Functional dependence, monotonicity, equations, linear regression, iterative processes, sequences, geometric series, limits

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

Heymann 1996, Huber 1997

Market equilibrium

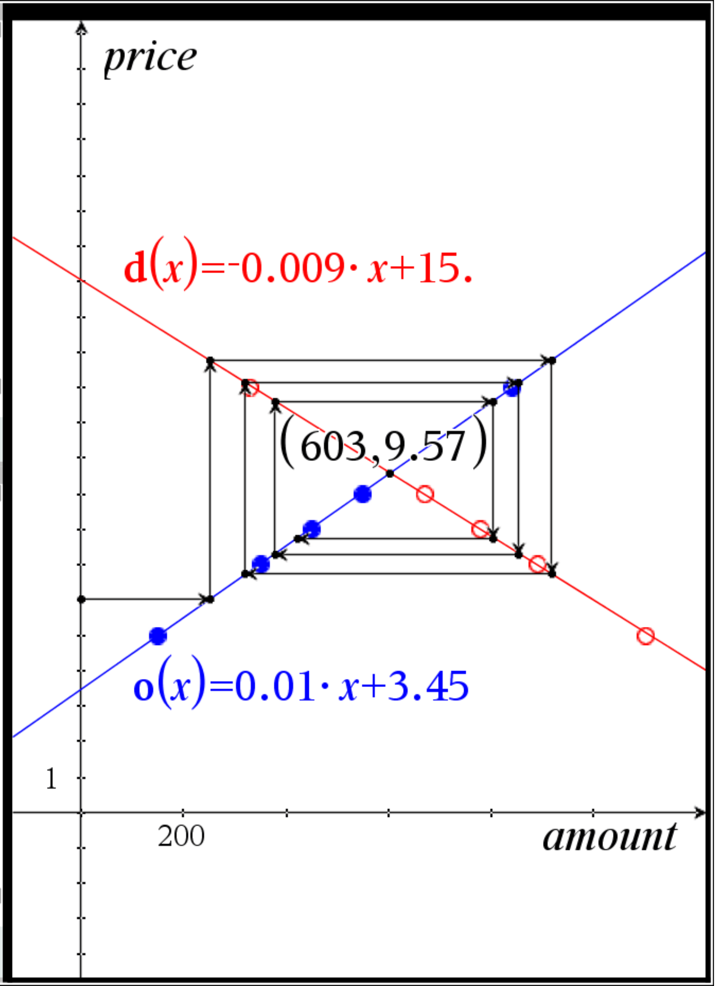
- Task: Determine the market price and the equilibrium quantity for the given relation between offer and demand of a single good!

	A price	B offer	C demand	D	E
=					
1	5	150	1100		
2	7	350	890		
3	8	450	780		
4	9	550	670		
5	12	840	330		

C1 1100

	A price_offer	B amount_offer	C price_demand
=			
1	6	251.098	12.7609
2	12.7609	917.667	6.71568
3	6.71568	321.658	12.121
4	12.121	854.576	7.28786
5	7.28786	378.07	11.6094
6	11.6094	804.135	7.74531

B1 =exp▸list(solve(o(x)=a1,x),{x})[1 1]

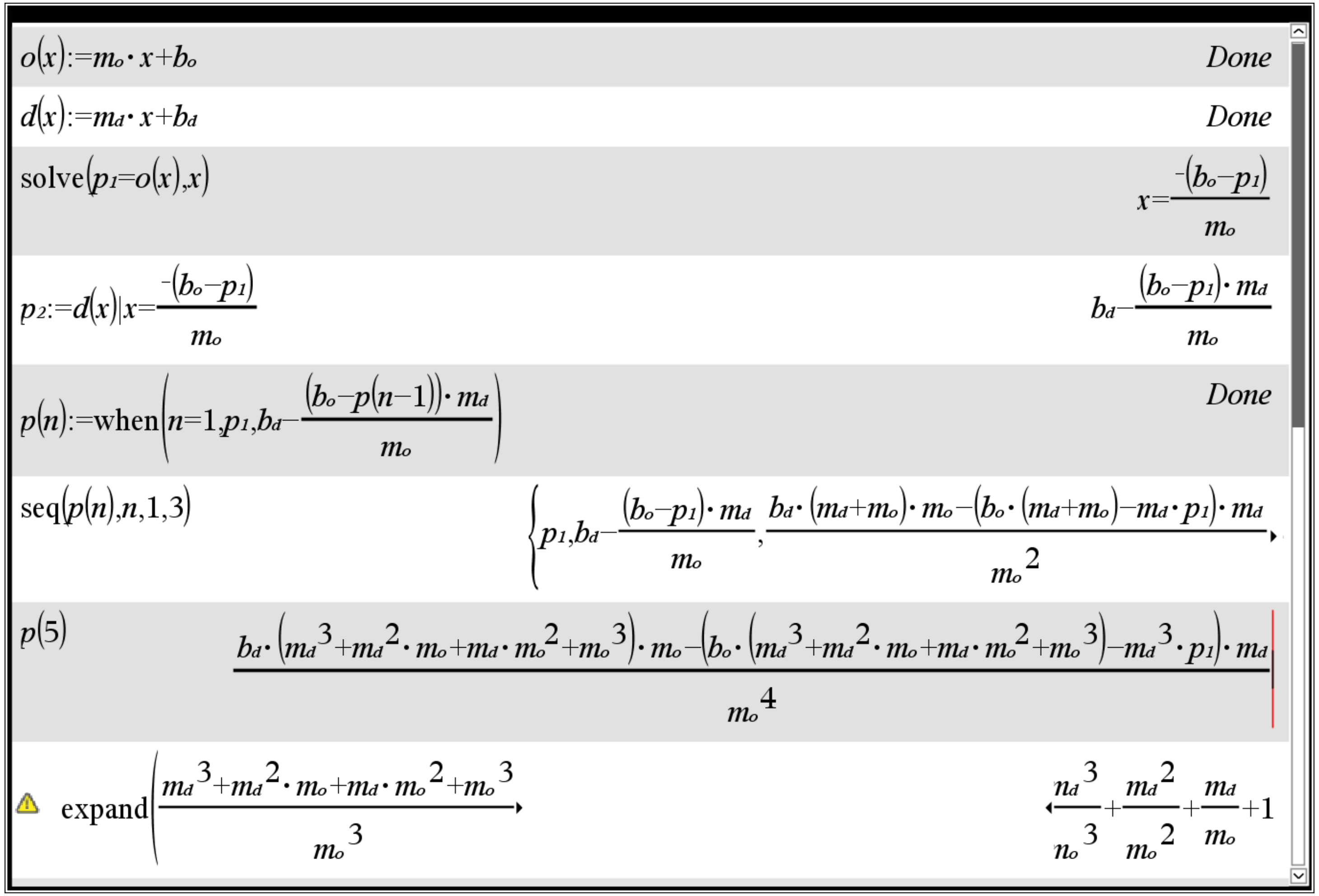


Tinhof et al. 2011

PÄDAGOGISCHE HOCHSCHULE SALZBURG

Algebraization

- Find criteria for the parameters $m_o, b_o, m_d,\ b_d$ in order to determine when the process of price development converges or not!
- Symbolic computation
  - Solve equations with parameters
  - Substitute
  - Find implicit and explicit representation of a sequence (seq, expand)



PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

Algebraization

- Calculation of series
- $|m_o| < |m_d|$
  process converges
  market is stable
- $|m_o| \geq |m_d|$
  process diverges
  market is instable

  price- amount-fluctuations do not lead to a market equlibirium (Cobweb Theory)

$$a(n):=a_1\cdot\left(\frac{m_o}{m_d}\right)^{n-1}-\frac{b_d-b_o}{m_d}\cdot\sum_{i=0}^{n-2}\left(\left(\frac{m_o}{m_d}\right)^{i}\right) \qquad \text{Done}$$

$$\sum_{i=0}^{n-2}\left(\left(\frac{m_o}{m_d}\right)^{i}\right) \qquad \frac{m_d}{m_d-m_o}-\frac{\left(\frac{m_o}{m_d}\right)^{n}\cdot m_d^{2}}{(m_d-m_o)\cdot m_o}$$

$$a(n):=a_1\cdot\left(\frac{m_o}{m_d}\right)^{n-1}-\frac{b_d-b_o}{m_d}\cdot\left(\frac{m_d}{m_d-m_o}-\frac{\left(\frac{m_o}{m_d}\right)^{n}\cdot m_d^{2}}{(m_d-m_o)\cdot m_o}\right. \blacktriangleright \qquad \text{Done}$$

$$0-\frac{b_d-b_o}{m_d}\cdot\frac{m_d}{m_d-m_o} \qquad \frac{-(b_d-b_o)}{m_d-m_o}$$

$$\frac{-(b_d-b_o)}{m_d-m_o}\Big|b_o=3.45316 \text{ and } b_d=15.0381 \text{ and } m_o=0.010143 \text{ and } m_d=-0.009069 \qquad 603.005$$

$$\text{solve}(o(x)=d(x),x) \qquad x=\frac{-(b_d-b_o)}{m_d-m_o}$$

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

Beutel 2010

Résumé

CAS can help to…

…draw attention to the process of problem-solving

- Formulate, Manipulate, Interpret

…develope strategic knowledge

- Substitute, equivalence of mathematical expressions

…support the process of algorithmic thinking

…illustrate abstract concepts (inference rules)

…describe aspects of our world (algebraization)

PÄDAGOGISCHE HOCHSCHULE SALZBURG Stefan Zweig

References

Beutel, Jörg (2010): Mikroökonomie. München [u.a.]: Oldenbourg.

Fuchs, Karl (1988): Erfahrungen und Gedanken zu Computern im Unterricht. In: *Journal für Mathematik-Didaktik* 9 (2/3), S. 247–256.

Fuchs, Karl; Plangg, Simon (2018): Computer Algebra Systeme in der Lehrer(innen)bildung. Münster: WTM Verlag für wissenschaftliche Texte und Medien (Scripta didactica mathematica, 5).

Heymann, H. W. (1996): Allgemeinbildung und Mathematik. Weinheim, Basel: Beltz Verlag.

Huber, Ludwig (1997): Organisationsformen des fächerübergreifenden Unterrichts. In: Werner Emler (Hg.): Ansätze zum fächerübergreifenden Unterricht in der gymnasialen Oberstufe: Lernen über Differenzen. Tagungsdokumentation und Anregungen für die Praxis. Bönen: Verlag für Schule und Weiterbildung, S. 53–75.

Hubwieser, Peter (2007): Didaktik der Informatik. 3. Aufl. Berlin [u.a.]: Springer (Springer-Lehrbuch).

Kortenkamp, Ulrich; Lambert, Anselm (2015): Wenn …, dann … bis … Algorithmisches Denken (nicht nur) im Mathematikunterricht. In: *mathematik lehren* 188, S. 2–9.

Profke, Lothar: Veranschaulichungen…nicht nur Visualisieren. In: Hermann Kautschitsch (Hg.): Anschauliche und experimentelle Mathematik II. Wien: Hölder-Pichler-Tempsky [u.a.] (Schriftenreihe Didaktik der Mathematik, 22), S. 13–30.

PÄDAGOGISCHE HOCHSCHULE SALZBURG

References

Schweiger, Fritz (2010): Fundamentale Ideen. Aachen: Shaker (Schriften zur Didaktik der Mathematik und Informatik an der Universität Salzburg, 3).

Siller, Hans-Stefan (2009): Der Begriff „Modellbilden" in der Mathematik- bzw. Informatikdidaktik. In: Michael Neubrand (Hg.): Beiträge zum Mathematikunterricht. 43. Jahrestagung der Gesellschaft für Didaktik der Mathematik. Oldenburg, 02. bis 06. März 2009. Münster: WTM Verlag für wissenschaftliche Texte und Medien.

Tinhof, Friedrich; Fischer, Wolfgang; Girlinger, Helmut; Paul, Markus; Schneider, Gerold (2011): Mathematik IV HAK/LW. Linz: Trauner Verlag.

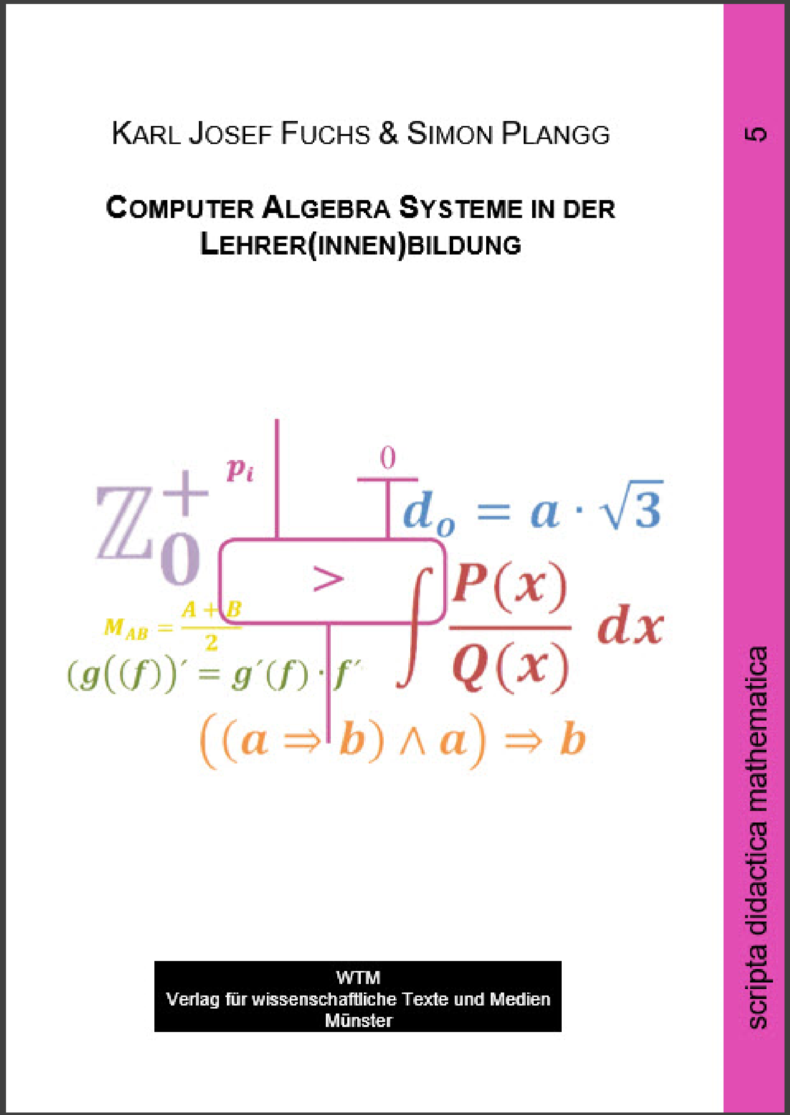
Weber, Agnes (2007): Problem-based learning. 2. Aufl. Bern: hep.

Wittmann, Erich (1981): Grundfragen des Mathematikunterrichts. 6. Aufl. Braunschweig: Vieweg.

Zimmermann, Bernd (1991): Ziele, Beispiele und Rahmenbedingungen problemorientierten Mathematikunterrichts. In: Bernd Zimmermann (Hg.): Problemorientierter Mathematikunterricht. Bad Salzdetfurth: Franzbecker, S. 9–36.

Thank you for your attention!

- Karl Josef Fuchs
  University of Salzburg
  KarlJosef.FUCHS@sbg.ac.at
- Simon Plangg
  College of Teacher Education Salzburg
  simon.plangg@phsalzburg.at



PÄDAGOGISCHE HOCHSCHULE SALZBURG